Sparse Recovery using Smoothed ℓ^0 (SL0): Convergence Analysis

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***Abstract*—Finding the sparse solution of an underdetermined system of linear equations has many applications, especially, it is used in Compressed Sensing (CS), Sparse Component Analysis (SCA), and sparse decomposition of signals on overcomplete dictionaries. We have recently proposed a fast algorithm, called Smoothed ℓ^0 (SL0), for this task. Contrary to many other sparse recovery algorithms, SL0 is not based on minimizing the ℓ^1 norm, but it tries to directly minimize the ℓ^0 norm of the solution. The basic idea of SL0 is optimizing a sequence of certain (continuous) cost functions approximating the ℓ^0 norm of a vector. However, in previous papers, we did not provide a complete convergence proof for SL0. In this paper, we study the convergence properties of SL0, and show that under a certain sparsity constraint in terms of Asymmetric Restricted Isometry Property (ARIP), and with a certain choice of parameters, the convergence of SL0 to the sparsest solution is guaranteed. Moreover, we study the complexity of SL0, and we show that whenever the dimension of the dictionary grows, the complexity of SL0 increases with the same order as Matching Pursuit (MP), which is one of the fastest existing sparse recovery methods, while contrary to MP, its convergence to the sparsest solution is guaranteed under certain conditions which are satisfied through the choice of parameters.**

***Index Terms*—Compressed Sensing (CS), Sparse Component Analysis (SCA), Sparse Decomposition, Atomic Decomposition, Over-complete Signal Representation, Sparse Source Separation.**

I. Introduction

SPARSE solution of an Underdetermined System of Linear Equations (USLE) has recently attracted the attention of many researchers from different viewpoints, because of its potential applications in many different problems. It is used, for example, in Compressed Sensing (CS) [1], [2], [3], underdetermined Sparse Component Analysis (SCA) and source separation [4], [5], [6], [7], atomic decomposition on overcomplete dictionaries [8], [9], decoding real field codes [10], etc.

Let $\mathbf{x}$ be a known $n \times 1$ vector and $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$ be a known $n \times m$ matrix with $m > n$, where $\mathbf{a}_i$'s denotes its columns. Then, we can seek the sparsest solution of the USLE $\mathbf{As} = \mathbf{x}$ given by

$$(P_0): \quad \min \|\mathbf{s}\|_0 \quad \text{s.t.} \quad \mathbf{As} = \mathbf{x}, \tag{1}$$

where $\|\cdot\|_0$ is simply the number of nonzero components (conventionally called the "ℓ^0" norm although it is not a true norm). In atomic decomposition viewpoint, $\mathbf{x}$ is a signal which is to be decomposed as a linear combination of the signals $\mathbf{a}_i$, $i = 1, \ldots, m$, where $\mathbf{a}_i$'s are called 'atoms', and $\mathbf{A}$ is called the 'dictionary' over which the signal is to be decomposed [11].

A system $\mathbf{A}$ is said [12] to satisfy *Unique Representation Property* (URP), if any $n \times n$ sub-matrix of $\mathbf{A}$ is invertible. It is known [12], [13], [14] that for any system satisfying URP, the solution to (1) is unique, that is if the a solution $\mathbf{s}_0$ satisfying $\|\mathbf{s}_0\|_0 < n/2$ exists, then any other solution $\mathbf{s}$ has $\|\mathbf{s}\|_0 > n/2$. Therefore, under URP assumption, we can talk about 'the sparsest solution'.

Solving (1) using a combinatorial search is NP-hard. Many alternative algorithms have been proposed to solve this problem. Two frequently used approaches are Matching Pursuit (MP) [11] and Basis Pursuit (BP) [8], which have many variants. MP is a fast algorithm but it cannot be guaranteed to find the sparsest solution. BP is based on replacing ℓ^0 with the ℓ^1 norm which can be minimized using Linear Programming techniques. BP is computationally more complex than MP, but it can find the sparsest solution with high probability, provided this solution is sufficiently sparse [13], [14], [2], [15].

In [16] and [17], we proposed an algorithm for solving (1), called Smoothed ℓ^0 (SL0), which provides a fast solution within a small Euclidean distance of the sparsest solution. The main idea was to approximate the ℓ^0 norm by a smooth function (hence the name "smoothed ℓ^0"). More precisely, $\|\mathbf{s}\|_0$ is approximated by a continuous function[1] $m - F_\sigma(\mathbf{s})$, where σ determines the quality of approximation: the larger σ, the smoother $F_\sigma(\cdot)$ but the worse the approximation to ℓ^0; and visa versa. Hence, the solution tends to the sparsest solution when $\sigma \to 0$. Therefore, the objective underlying SL0 is to maximize $F_\sigma(\mathbf{s})$ (subject to $\mathbf{As} = \mathbf{x}$) for some *very small* value of σ. However, for small values of σ, $F_\sigma(\cdot)$ has many local maxima and hence its maximization is not easy. Therefore, SL0 uses a Graduated Non-Convexity (GNC) [18] approach: It starts from a very large σ (for which there is no local maxima), and gradually decreases σ to zero. The

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[1]In this form, $F_\sigma(\mathbf{s})$ is an approximation to the number of 'zero's of $\mathbf{s}$, that is, $m - \|\mathbf{s}\|_0$.

- Initialization: Set $\hat{\mathbf{s}}_0 = \mathbf{A}^\dagger \mathbf{x}$. Choose a suitable decreasing sequence for σ: $[\sigma_1 \dots \sigma_J]$.
- For $j = 1, \dots, J$:
  1) Let $\sigma = \sigma_i$.
  2) Maximize $F_\sigma(\mathbf{s})$ subject to $\mathbf{As} = \mathbf{x}$, using L iterations of steepest ascent:
     - Initialization: $\mathbf{s} = \hat{\mathbf{s}}_{j-1}$.
     - For $\ell = 1, 2, \dots, L$
       a) Let $\mathbf{s} \leftarrow \mathbf{s} + (\mu\sigma^2)\nabla F_\sigma(\mathbf{s})$.
       b) Project $\mathbf{s}$ back onto the feasible set $\{\mathbf{s}|\mathbf{As} = \mathbf{x}\}$:
       $$\mathbf{s} \leftarrow \mathbf{s} - \mathbf{A}^\dagger(\mathbf{As} - \mathbf{x}).$$
  3) Set $\hat{\mathbf{s}}_j = \mathbf{s}$.
- Final answer is $\hat{\mathbf{s}} = \hat{\mathbf{s}}_J$.

Fig. 1. Basics of the SL0 algorithm [17]. $\mathbf{A}^\dagger$ stands for the Moore-Penrose pseudo inverse of $\mathbf{A}$ (*i.e.* $\mathbf{A}^\dagger \triangleq \mathbf{A}^T(\mathbf{AA}^T)^{-1}$).

maximum of $F_\sigma(\cdot)$ is used as a starting point to locate the maximum of $F_\sigma(\cdot)$ for the next (smaller) σ using a steepest ascent approach. Since the value of σ has only slightly decreased, the maximizer of $F_\sigma(\cdot)$ for this new σ is not too far from the maximizer of $F_\sigma(\cdot)$ for the previous (larger) σ, and hence it is hoped that it does not get trapped into a local maximum. Figure 1 shows the basics of SL0 algorithm[2].

From Fig. 1, SL0 consists of two loops: the 'outer' loop is the loop in which σ is decreased, and the 'inner' loop is the one in which $F_\sigma(\mathbf{s})$ is iteratively maximized (subject to $\mathbf{As} = \mathbf{x}$) for the *fixed* choice of σ. In [17], we prove that *if the inner loop does not get trapped in a local maximum, our solution will converge to the solution of (1)* as $\sigma \to 0$ in the outer loop. In other words, if σ is decreased so gradually that the GNC approach works and we have avoided local maxima in the inner loop, then our method will produce the desired results.

However, a complete convergence analysis of SL0, as well as the choice of SL0 parameters to guarantee avoiding local maxima in the inner loops remained to be shown. In particular, we want to know 1) the rate of decreasing of σ, 2) how many times we have to repeat the inner loop (the value of L), and 3) how to choose μ in Fig. 1. In this paper, we present a complete convergence analysis of SL0 for both noiseless and noisy cases, and we present parameter settings that guarantee SL0 convergence to the solution of (1). In contrast to exponential family of functions used for approximating the ℓ^0 norm in [17], the analysis here uses a family of spline functions for this aim.

Note that, in practice, the values of SL0 parameters that guarantee the convergence to the solution of (1) are not necessarily 'good' values. These values provide a theoretical support for the SL0 algorithm, but they are often excessively pessimistic and result in slower convergence of the algorithm compared to a *typical* behavior (see also Section VI of [9]).

[2]Two other points in Fig. 1 are: 1) The initial guess for the sparsest solution is the minimum ℓ^2 norm solution of $\mathbf{As} = \mathbf{x}$, which corresponds [17] to the maximizer of $F_\sigma(\mathbf{s})$ where $\sigma \to \infty$, and 2) The step-size of the steepest ascent is decreased proportional to σ^2 [17].

A. Restricted Isometry and Overview of the Results

The analysis is developed here using the Asymmetric Restricted Isometry Constants (ARICs) [19], [20], [21], [22], in order to relate our work to ℓ^1-minimization. The asymmetric k-restricted constants $\delta_k^{\min}$ and $\delta_k^{\max}$ are defined as the smallest nonnegative numbers satisfying

$$(1 - \delta_k^{\min})\|\mathbf{s}\|_2^2 \leq \|\mathbf{As}\|_2^2 \leq (1 + \delta_k^{\max})\|\mathbf{s}\|_2^2 \tag{2}$$

for any $\mathbf{s} \in \mathbb{R}^m$ with $\|\mathbf{s}\|_0 \leq k$.

Let $\mathbf{s}_0$ be the solution of (1) and $\|\mathbf{s}_0\|_0 = k$. We show that SL0 recovers this solution provided that

$$\alpha\delta_{\lceil 2k\alpha \rceil}^{\min} + \|\mathbf{A}\|_2 \leq \alpha \tag{3}$$

for any $\alpha > 1$, in which $\|\mathbf{A}\|_2$ denotes the Euclidean norm of $\mathbf{A}$, and $\lceil 2k\alpha \rceil$ denotes the nearest integer greater than or equal to $2k\alpha$. More precisely, we derive a family of sufficient conditions for the performance of SL0 that depend on parameter α.

The ARICs are easy to calculate exactly for small scale systems, but the complexity grows exponentially as the scale grows. In fact, the value of ARICs depends on singular values of sub-matrices of the matrix $\mathbf{A}$. Then, using the results of [23], [24], [25], [19], [20], [21], we analyze the behavior of SL0 for large Gaussian random dictionaries. To achieve bounds similar to the existing ones for ℓ^1 minimization methods, we use a popular result in Random Matrix Theory [26], [27], to derive Corollary 4 of Section III which can be viewed as SL0 counterpart of Theorem 3.1 of [28]. Specifically, we identify $\rho(\alpha) > 0$, for any $0 < \alpha \leq 1$, such that for large scales[3] satisfying $n/m \to \alpha$ and $m \to \infty$, SL0 can recover any sparse solution $\mathbf{s}$ with $\|\mathbf{s}\|_0 < \rho(\alpha)m$ from a (possibly noisy) measurement $\mathbf{x}$.

One of the bottlenecks of Compressed Sensing methods for handling large scale systems is the decoding complexity (see [10] for the definition of encoding and decoding in compressed sensing context). In BP, decoding complexity is known to be m^3 [10], [23], or $m^{1.5}n^2$ for the cases where n is much smaller than m [29], [30]. The coding complexity is mn. MP method has the smallest possible complexities for both encoding and decoding, which is mn [31]. For certain classes of systems, the complexity can be further reduced to $m \log m$ [32]. In this paper, we will see (in Section VI-C) that the coding and decoding complexities of SL0 are similar to that of MP.

Since (1) is NP-hard, one may wonder that proving convergence of SL0 (with a complexity growing in quadratic with scale) means that NP = P. This is not the case. Note that in BP, too, the guarantee that BP will find the solution of (1) does not mean that NP = P, because such a guarantee only exists in the case of a very sparse solution. Our analysis possesses a similar limitation, too.

The paper is organized as follows. In section II, assuming that the internal loop of Fig. 1 exactly follows the steepest ascent trajectory (in other words, we ignore the effect of μ and L, or implicitly assume that $\mu \to 0$ and $L \to \infty$), we analyze the convergence of the resultant (i.e. asymptotic)

[3]By scale we mean the number of rows, n, and the number of columns, m, of the dictionary.

SL0. Indeed, in this section, Theorem 2 proposes a geometric σ sequence which guarantees the local concavity of cost functions and the convergence of the internal loop of SL0 to the true maximizer of F_σ, and hence the convergence of asymptotic SL0 to the sparsest solution. This sequence depends on the ARIP constants of the dictionary, which are not easy to calculate. Hence, in Section III, we discuss the behavior of asymptotic SL0 in the case of large random Gaussian dictionaries. Corollary 4 of this section corresponds Donoho's results for ℓ^1 minimization, Theorem 3.1 of [28]. In Section IV, we consider the effect of no ideal μ, that is, where the internal loop does not follow exactly the steepest ascent trajectory, and makes discrete jumps in the steepest ascent direction. We provide a choice for μ which guaranties stability of the internal loop and convergence to the maximizer as $L \to \infty$. Then, after a discussion on the noisy case in Section V, we derive a (finite) value for L in Section VI which guaranties the convergence of SL0 to the sparsest solution. This completes our convergence analysis of SL0. Further in Section VI (Theorem 7), we study the complexity of SL0 and prove that it is of order $O(m^2)$, that is, the same as for MP, which is the fastest known algorithm in the field. Finally we address multiple sparse solution recovery with SL0 and show that the order of complexity of SL0 can be reduced to $O(m^{1.376})$ in this case.

II. Convergence Analysis in noiseless case

A. Basic Definitions

In [17], we first choose a continuous function f_σ that asymptotically approximates a Kronecker delta:

$$\lim_{\sigma\to 0} f_\sigma(s) = \begin{cases} 1 & ;\text{if } s = 0 \\ 0 & ;\text{if } s \neq 0 \end{cases}, \tag{4}$$

and use it to approximate $\|\mathbf{s}\|_0$ by $m - F_\sigma(\mathbf{s})$ where $F_\sigma(\mathbf{s}) \triangleq \sum_{i=1}^m f_\sigma(s_i)$. Then, it is shown that under some mild conditions on $f_\sigma(\cdot)$, maximizing $F_\sigma(\mathbf{s})$ on $\mathbf{As} = \mathbf{x}$ for a small σ, using a GNC approach, will recover the sparse solution. To avoid being trapped into local maxima, one may wish to design a continuous concave function f_σ that can asymptotically approximate a Kronecker delta, but, taking into account the shape of any approximation to the Kronecker delta, this is not possible. However, we note that even for non-concave continuous functions, if the function is concave in the vicinity of the global maximum then by starting from any point sufficiently close the global maximum, steepest ascent will converge to the global maximum. In this section we investigate conditions under which F_σ subject to $\mathbf{As} = \mathbf{x}$ is concave near the global maximum, and how these can be used in designing a sequence of σ that forces SL0 to converge to the global maximum.

Remark 1. Without loss of generality, we assume that the rows of $\mathbf{A}$ are orthonormal, *i.e.* $\mathbf{AA}^T = \mathbf{I}_n$, where $\mathbf{I}_n$ stands for the $n\times n$ identity matrix. In effect, if the rows of $\mathbf{A}$ are not orthonormal, performing a Gram-Schmidt orthonormalization on the rows of $\mathbf{A}$ (and doing the corresponding operations on $\mathbf{x}$, too) gives rise to an equivalent system of equations with the same set of solutions and with orthonormal rows of its dictionary.

Moreover, for any matrix $\mathbf{A}$ with orthonormal rows, by expanding the set of rows of $\mathbf{A}$, one can find a matrix $\mathbf{D} \in \mathbb{R}^{(m-n)\times m}$ such that $\mathbf{Q} = [\mathbf{A}^T, \mathbf{D}^T]^T$ is orthonormal. We note then that:

$$\begin{cases} \mathbf{AA}^T = \mathbf{I}_n \\ \mathbf{DD}^T = \mathbf{I}_{m-n} \\ \mathbf{AD}^T = \mathbf{0} \\ \mathbf{A}^T\mathbf{A} + \mathbf{D}^T\mathbf{D} = \mathbf{I}_m \end{cases}. \tag{5}$$

The rows of the matrix $\mathbf{D}$ are an orthonormal basis for the null-space of $\mathbf{A}$. Moreover, for any $\mathbf{s}$ satisfying $\mathbf{As} = \mathbf{0}$ we have

$$\|\mathbf{Ds}\| = \|\mathbf{Qs}\| = \|\mathbf{s}\|, \tag{6}$$

where, throughout the paper, $\|\cdot\|$ stands for the ℓ^2 norm of a vector.

Definition 1: Let $\pi_i : \mathbb{R}^m \mapsto \mathbb{R}$ be the projection of $\mathbf{s} = [s_1, \cdots, s_m]^T$ onto the ith axis, *i.e.* $\pi_i(\mathbf{s}) = s_i$. Moreover, let $\pi_I(\mathbf{s}) = (s_{i_1}, \cdots, s_{i_r})^T$ for $I = \{i_1 < i_2 < \cdots < i_r\} \subseteq \{1\cdots m\}$. Also let $I^c = \{1\cdots m\} - I$.

Example. For $\mathbf{s} = (2, 3, 4, 7)^T$ and $I = \{1, 3\}$, we have $\pi_3(\mathbf{s}) = 4$, $\pi_I(\mathbf{s}) = (2, 4)^T$, and $\pi_{I^c}(\mathbf{s}) = (3, 7)^T$.

Definition 2: For the matrix $\mathbf{A}$ we define:

$$\begin{aligned}\gamma_{\mathbf{A}}(n_0) &\triangleq \max_{|I|\le n_0} \max_{\mathbf{As}=\mathbf{0}} \frac{\|\pi_I(\mathbf{s})\|^2}{\|\pi_{I^c}(\mathbf{s})\|^2} \\ &= \max_{|I|\le n_0} \max_{\mathbf{As}=\mathbf{0}} \frac{\|\mathbf{s}\|^2 - \|\pi_{I^c}(\mathbf{s})\|^2}{\|\pi_{I^c}(\mathbf{s})\|^2} \\ &= \max_{|I|\le n_0} \max_{\mathbf{As}=\mathbf{0}} \frac{\|\mathbf{s}\|^2}{\|\pi_{I^c}(\mathbf{s})\|^2} - 1,\end{aligned} \tag{7}$$

where $|I|$ represents the cardinality of I. We will use $\gamma(n_0) = \gamma_{\mathbf{A}}(n_0)$ notation whenever there is no ambiguity about the matrix $\mathbf{A}$.

Remark 2. Let null$(\mathbf{A}) = \{\mathbf{s} \in \mathbb{R}^m | \mathbf{As} = \mathbf{0}\}$ denote the null space of $\mathbf{A}$. Then for any $\mathbf{s} \in$ null$(\mathbf{A})$:

$$\mathbf{As} = 0 \Rightarrow \mathbf{A}_I\mathbf{s}_I + \mathbf{A}_{I^c}\mathbf{s}_{I^c} = 0 \Rightarrow \|\mathbf{A}_I\mathbf{s}_I\| = \|\mathbf{A}_{I^c}\mathbf{s}_{I^c}\|, \tag{8}$$

where $\mathbf{A}_I$ and $\mathbf{A}_{I^c}$ are sub-matrices of $\mathbf{A}$ containing columns indexed by I and I^c, respectively, $\mathbf{s}_I \triangleq \pi_I(\mathbf{s})$ and $\mathbf{s}_{I^c} \triangleq \pi_{I^c}(\mathbf{s})$. Now let $\sigma_{\min}(\cdot)$ and $\sigma_{\max}(\cdot)$ stand for the smallest and largest singular values of a matrix[4]. Then from (8) and

$$\begin{aligned}\|\mathbf{A}_I\mathbf{s}_I\| &\ge \sigma_{\min}(\mathbf{A}_I)\|\mathbf{s}_I\| \\ \|\mathbf{A}_{I^c}\mathbf{s}_{I^c}\| &\le \sigma_{\max}(\mathbf{A}_{I^c})\|\mathbf{s}_{I^c}\|\end{aligned} \tag{9}$$

we will have:

$$\gamma(n_0) \le \max_{|I|\le n_0} \frac{\sigma^2_{\max}(\mathbf{A}_{I^c})}{\sigma^2_{\min}(\mathbf{A}_I)}. \tag{10}$$

[4]While it is common in the literature to define singular values to be strictly positive, in this paper, we use the definition of Horn and Johnson [33, pp. 414-415], in which, the number of singular values of a $p \times q$ matrix $\mathbf{M}$ is fixed equal to $\min(p, q)$, and hence, the singular values of $\mathbf{M}$ are the square roots of *the* $\min(p, q)$ *largest eigenvalues of* $\mathbf{M}^H\mathbf{M}$ *(or* $\mathbf{MM}^H$*)*. Using this definition, a matrix can have zero singular values; where a zero singular value characterizes a non-full-rank matrix.

By a similar argument:

$$\gamma(n_0)+1 \le \max_{|I|\le n_0} \frac{\sigma^2_{\max}(\mathbf{A})}{\sigma^2_{\min}(\mathbf{A}_I)} = \frac{\|\mathbf{A}\|_2^2}{\min_{|I|\le n_0}\sigma^2_{\min}(\mathbf{A}_I)}. \quad (11)$$

where $\|\cdot\|_2$ denotes the spectral norm of a matrix, that is, its largest singular value.

Remark 3. $\gamma(n) < \infty$ as long as $\mathbf{A}$ satisfies the URP. Observe that for any subset $|I| \le n$, we have $\sigma^2_{\max}(\mathbf{A}_{I^c}) < \infty$. When $\mathbf{A}$ has the URP, the columns of $\mathbf{A}_I$ are linearly independent as long as $|I| \le n$, and hence $\sigma^2_{\min}(\mathbf{A}_I) > 0$. Then (10) implies that $\gamma(n)$ is finite.

Remark 4. $\gamma(n_0)$ is clearly an increasing function of n_0.

Remark 5. Our definition of $\gamma(n_0)$ in (7) relates to the lower ARIC defined in (2). From (11) it is easy to see that for the ARIC δ_k^{min} satisfying (2),

$$\gamma(n_0)+1 \le \frac{\|\mathbf{A}\|_2^2}{1-\delta^{\min}_{n_0}}. \quad (12)$$

Considering the existing upper bounds on the ARIP constants [20], it is straight forward to find upper bound on $\gamma(n_0)$. We discuss the upper bound on $\gamma(n_0)$ in section III.

Remark 6. For any $n\times n$ nonsingular matrix $\mathbf{Q}$, the null spaces of $\mathbf{A}$ and $\mathbf{QA}$ are equal, *i.e.* $\{\mathbf{s}|\mathbf{As}=\mathbf{0}\} = \{\mathbf{s}|\mathbf{QAs}=\mathbf{0}\}$. Therefore, $\gamma_{\mathbf{A}}(n_0)=\gamma_{\mathbf{QA}}(n_0)$ for any value of n_0.

Remark 7. Gram Schmidt orthonormalization involves left side multiplication by a nonsingular matrix. Therefore, it does not change the value of γ.

In [17], we had used a family of Gaussian functions to approximate the ℓ^0 norm. In this paper we use quadratic splines instead. The second order derivative of these splines is easy to manipulate and this simplifies our convergence analysis.

Definition 3: Let $f_\gamma : \mathbb{R} \mapsto \mathbb{R}$ denote a quadratic spline with knots at $\{+1,-1,1+\gamma,-1-\gamma\}$, that is:

$$f_\gamma(s) \triangleq \begin{cases} 1-s^2/(1+\gamma) & ;\text{if } |s|\le 1 \\ (|s|-\gamma-1)^2/(\gamma^2+\gamma) & ;\text{if } 1\le |s| \le 1+\gamma \\ 0 & ;\text{if } |s|\ge 1+\gamma \end{cases}. \quad (13)$$

We also define

$$f_{\gamma,\sigma}(s) \triangleq f_\gamma(s/\sigma) \quad (14)$$

and

$$F_{\gamma,\sigma}(\mathbf{s}) \triangleq \sum_{i=1}^m f_{\gamma,\sigma}(s). \quad (15)$$

In the rest of this paper, we use the notation $F_\gamma = F_{\gamma,1}$. We also use $F_\sigma = F_{\gamma,\sigma}$ whenever there is no ambiguity about γ.

Remark 8. f_γ and f'_γ are both continuous, so that

$$f'_\gamma(s) = \begin{cases} -2s/(1+\gamma) & ;\text{if } |s|\le 1 \\ 2s/(\gamma^2+\gamma)-2/\gamma & ;\text{if } 1\le s\le 1+\gamma \\ 2s/(\gamma^2+\gamma)+2/\gamma & ;\text{if } -1-\gamma\le s\le -1 \\ 0 & ;\text{if } |s|\ge 1+\gamma \end{cases}, \quad (16)$$

and

$$f''_\gamma(s) = \begin{cases} -2/(\gamma+1) & ;\text{if } |s|\le 1 \\ 2/(\gamma^2+\gamma) & ;\text{if } 1\le |s|\le 1+\gamma \\ 0 & ;\text{if } |s|\ge 1+\gamma \end{cases}. \quad (17)$$

Definition 4: By $\|\mathbf{s}\|_{0,\sigma}$, we mean the number of elements of $\mathbf{s}$ which have absolute values greater than σ. In other words, $\|\mathbf{s}\|_{0,\sigma}$ denotes the ℓ^0 norm of a clipped version of $\mathbf{s}$, in which, the components with absolute values less than or equal to α have been clipped to zero.

B. *Local concavity of the cost functions*

In this subsection, we show that $F=F_{\gamma,\sigma}$ defined in (15), with $\gamma=\gamma(n_0)$, $n_0\le n$, and restricted to a certain subset of $\mathcal{S}_{\mathbf{x}} \triangleq \{\mathbf{s}\in\mathbb{R}^m|\mathbf{As}=\mathbf{x}\}$, is concave. Then, we show that this subset includes all points for which $F > n_0/(1+\gamma)$.

Lemma 1: Lets denote $F=F_{\gamma,\sigma}$, where $\gamma=\gamma(n_0)$ for $n_0\le n$, and have $\mathbf{A}$ satisfy the URP. Let $\mathcal{S}_{\mathbf{x}} \triangleq \{\mathbf{s}\in\mathbb{R}^m|\mathbf{As}=\mathbf{x}\}$ and $\mathcal{C}$ be the subset of $\mathcal{S}_{\mathbf{x}}$ consisting of those solutions that have at most n_0 elements with absolute values greater than σ, that is:

$$\mathcal{C} \triangleq \{\mathbf{s}\in\mathcal{S}_{\mathbf{x}}|\, \|\mathbf{s}\|_{0,\sigma}\le n_0\}. \quad (18)$$

Then the Hessian matrix of $F|_{\mathcal{C}}$, where $F|_{\mathcal{C}}$ denotes the restriction of F on $\mathcal{C}$, is negative semi-definite.

Proof: Let the linear transformation $T:\mathbb{R}^{m-n}\mapsto\mathcal{S}_{\mathbf{x}}$ defined by $\mathbf{s}=T(\mathbf{v}) \triangleq \mathbf{D}^T\mathbf{v}+\mathbf{A}^T\mathbf{x}$ for a constant $\mathbf{x}$. T is clearly a linear isomorphism. Hence, instead of showing that the Hessian of $F|_{\mathcal{C}}$ is negative semi-definite, we just need to show that the Hessian of G is negative semi-definite on $T^{-1}(\mathcal{C})\subseteq\mathbb{R}^{m-n}$, where $G=F\circ T$.

Assume $\mathbf{s}\in\mathcal{C}$. Clearly

$$\mathbf{H}_G(\mathbf{v}) = \mathbf{D}\mathbf{H}_F(\mathbf{s})\mathbf{D}^T,$$

where $\mathbf{v}=T^{-1}(\mathbf{s})$ and

$$\begin{aligned}\mathbf{H}_F(\mathbf{s}) &= \operatorname{diag}\big(f''_{\gamma,\sigma}(s_1),\dots,f''_{\gamma,\sigma}(s_m)\big)\\ &= \frac{1}{\sigma^2}\operatorname{diag}\big(f''_\gamma(s_1/\sigma),\dots,f''_\gamma(s_m/\sigma)\big).\end{aligned}$$

Let I be the set of indexes of those elements of $\mathbf{s}\in\mathcal{C}$ that have absolute values greater than σ. From the definition of $\mathcal{C}$, $|I|\le n_0$. To prove that $\mathbf{H}_G(\mathbf{v})$ is negative semi-definite, we have to show that $\mathbf{u}^T\mathbf{D}^T\mathbf{H}_F(\mathbf{s})\mathbf{Du}\le 0$ for all $\mathbf{u}\in\mathbb{R}^m$. Defining $\mathbf{w}\triangleq\mathbf{Du}$, we have $\mathbf{Aw}=\mathbf{ADu}=\mathbf{0}$ and, therefore, $\mathbf{w}\in$ null$(\mathbf{A})$. Next we show that $\mathbf{w}^T\mathbf{H}_F(\mathbf{s})\mathbf{w}\le 0$ for all $\mathbf{w}\in$ null$(\mathbf{A})$. We write:

$$\begin{aligned}\mathbf{w}^T\mathbf{H}_F(\mathbf{s})\mathbf{w} &= \frac{1}{\sigma^2}\sum_{i=1}^m f''_\gamma(s_i/\sigma){w_i}^2\\ &= \frac{1}{\sigma^2}\sum_{i\in I} f''_\gamma(s_i/\sigma){w_i}^2 + \frac{1}{\sigma^2}\sum_{i\notin I} f''_\gamma(s_i/\sigma){w_i}^2.\end{aligned} \quad (19)$$

By setting $\mathbf{w}_I$ and $\mathbf{w}_{I^c}$ equal to the sub-vectors of $\mathbf{w}$ indexed by I and I^c, from (7) we have

$$\frac{\|\mathbf{w}_I\|^2}{\|\mathbf{w}_{I^c}\|^2} = \frac{\|\pi_I(\mathbf{w})\|^2}{\|\pi_{I^c}(\mathbf{w})\|_2^2} \le \gamma(|I|) \le \gamma(n_0)=\gamma, \quad (20)$$

and hence, using (17):

$$\mathbf{w}^T\mathbf{H}_F(\mathbf{s})\mathbf{w} \le -\frac{2}{1+\gamma}\frac{\|\mathbf{w}_{I^c}\|^2}{\sigma^2} + \frac{2}{\gamma^2+\gamma}\frac{\|\mathbf{w}_I\|^2}{\sigma^2} \le 0,$$

which completes the proof. ∎

Corollary 1: Under the conditions of Lemma 1, $F = F_{\gamma,\sigma}$ is concave at every $\mathbf{s} \in \mathcal{B}$, where $\mathcal{B} \triangleq \{\mathbf{s} \in \mathcal{S}_{\mathbf{x}} | F(\mathbf{s}) \geq m - n_0/(1+\gamma)\}$. Moreover, the region $\mathcal{A} \triangleq \{\mathbf{s} \in \mathcal{S}_{\mathbf{x}} | F(\mathbf{s}) \geq m - n_0/(2+2\gamma)\} \subseteq \mathcal{B}$ is convex.

Proof: To prove the first part we show that $\mathcal{B} \subseteq \mathcal{C}$, where $\mathcal{C}$ is defined by 18. Let $\mathbf{s} \in \mathcal{B}$ and $I \triangleq \{1 \leq i \leq m \,|\, |s_i| > \sigma\}$. Then $\|\mathbf{s}\|_{0,\sigma} = |I|$, and hence, to prove $\mathbf{s} \in \mathcal{C}$ we have to show that $|I| \leq n_0$. We write:

$$\forall i \in I : |s_i| \geq \sigma \Rightarrow 1 - f(s_i) \geq 1/(1+\gamma) \tag{21}$$

$$\begin{aligned} \mathbf{s} \in \mathcal{B} \Rightarrow \frac{n_0}{1+\gamma} \geq m - F(\mathbf{s}) &= \sum_{i=1}^{m} \{1 - f(s_i)\} \\ &\geq \sum_{i \in I} \{1 - f(s_i)\}. \end{aligned} \tag{22}$$

Substituting (21) in (22), we obtain $n_0/(1+\gamma) \geq |I|/(1+\gamma)$, which completes the proof of the first part.

To prove the second part, we consider $\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{A}$. By definition, at most $\frac{n_0}{2}$ elements of $\mathbf{s}_1$ and $\mathbf{s}_2$ can be greater than σ. Hence, if we define $\mathbf{s}(t) = (1-t)\mathbf{s}_1 + t\mathbf{s}_2$, for $0 \leq t \leq 1$, at most n_0 elements of $\mathbf{s}(t)$ can have absolute values greater than σ. We know $\dot{\mathbf{s}}(t) = \mathbf{s}_2 - \mathbf{s}_1 \in \text{null}(\mathbf{A})$, and the hessian of F is negative semi-definite on null($\mathbf{A}$) according to Lemma 1. Hence, if we define $h(t) = F(\mathbf{s}(t))$, we obtain:

$$\ddot{h} = \dot{\mathbf{s}}^T \mathbf{H}_F \dot{\mathbf{s}} + (\nabla F)^T \ddot{\mathbf{s}} = \dot{\mathbf{s}}^T \mathbf{H}_F \dot{\mathbf{s}} \leq 0\cdot$$

Hence, h is concave on the $[0,1]$ interval, and for any $0 \leq t \leq 1$, we have

$$F(\mathbf{s}(t)) = h(t) \geq t \cdot h(1) + (1-t) \cdot h(0) \geq m - n_0/(2+2\gamma)$$

. This implies that $\mathbf{s}(t) \in \mathcal{A}$, hence $\mathcal{A}$ is convex. ∎

Corollary 2: Under the conditions of Lemma 1 and the assumption that there exists a sparse solution $\mathbf{s}_0$ satisfying $k \triangleq \|\mathbf{s}_0\|_0 \leq n_0/(2+2\gamma)$, by starting from any $\hat{\mathbf{s}}$ satisfying $F(\hat{\mathbf{s}}) \geq m - n_0/(2+2\gamma)$ and moving on the steepest ascent trajectory restricted to $\mathcal{S}_{\mathbf{x}}$, we reach the global maximum $\mathbf{s}_*$ of $F|_{\mathcal{S}_{\mathbf{x}}}$, satisfying $F(\mathbf{s}_*) \geq m - k$. More precisely, the solution of the differential equation

$$\begin{cases} \dot{\boldsymbol{\alpha}}(t) = \nabla F|_{\mathcal{S}_{\mathbf{x}}} \\ \boldsymbol{\alpha}(0) = \hat{\mathbf{s}} \end{cases} \tag{23}$$

satisfies

$$\lim_{t \to +\infty} \boldsymbol{\alpha}(t) = \mathbf{s}_* \,. \tag{24}$$

Proof: From Corollary 1 we know that $\mathcal{A} = \{\mathbf{s} | F(\mathbf{s}) \geq m - n_0/(2+2\gamma)\}$ is a convex region. By starting from any point in a convex region and moving on the steepest ascent trajectory of a function which is concave on that region, we achieve the global maximizer in that region. Therefore, the steepest ascent trajectory leads to the maximizer $\mathbf{s}_* \in \mathcal{A}$. Using the assumptions on sparse solution, we have $\mathbf{s}_0 \in \mathcal{A}$. Hence the maximizer clearly satisfies $F(\mathbf{s}_*) \geq F(\mathbf{s}_0) \geq m - k$. ∎

C. *The narrow variation property*

In this subsection, we introduce a notion of the narrow variation property, which states that whenever the values of F_σ at two points exceed a certain threshold, those two points are close to each other in the sense of the Euclidean distance between them being bounded by $O(m^{1/2}\gamma^{1/2}\sigma)$. Before stating Lemma 2, we repeat Theorem 1 from [17]. This theorem states that if for each value of σ we pick a point $\mathbf{s}_\sigma$ on $\mathcal{S}_{\mathbf{x}}$ such that $F_\sigma(\mathbf{s}_\sigma)$ is greater than a certain value $m - n + k$, then the sequence of these points converges to the sparsest solution as $\sigma \to 0$.

Theorem 1: Consider a family of univariate functions f_σ, indexed by σ, $\sigma \in \mathbb{R}^+$, satisfying the set of conditions:

1) $\lim_{\sigma \to 0} f_\sigma(s) = 0$; for all $s \neq 0$
2) $f_\sigma(0) = 1$; for all $\sigma \in \mathbb{R}^+$
3) $0 \leq f_\sigma(s) \leq 1$; for all $\sigma \in \mathbb{R}^+, s \in \mathbb{R}$
4) For each positive values of ν and α, there exists $\sigma_0 \in \mathbb{R}^+$ that satisfies:

$$|s| > \alpha \Rightarrow f_\sigma(s) < \nu \quad ; \text{for all } \sigma < \sigma_0. \tag{25}$$

Let $F_\sigma(\mathbf{s}) \triangleq \sum_{i=1}^{n} f_\sigma(s_i)$. Assume that $\mathbf{A}$ satisfies the URP, $\mathbf{s}_0 \in \mathcal{S}_{\mathbf{x}}$ satisfies $\|\mathbf{s}_0\|_0 = k \leq n/2$ and $\mathbf{s}_\sigma \in \mathcal{S}_{\mathbf{x}}$ satisfies $F_\sigma(\mathbf{s}_\sigma) \geq m - n + k$. Then

$$\lim_{\sigma \to 0} \mathbf{s}_\sigma = \mathbf{s}_0. \tag{26}$$

Remark 1. Note that the conditions on $\mathbf{A}$ in Lemma 1 are the same as in Theorem 1, and $f_{\gamma,\sigma}$ defined in (14) satisfies all the conditions 1 to 4 of Theorem 1, for any arbitrary value of γ.

The main idea of the following Lemma 2 (and its proof) is very similar to that of Theorem 1. We prove that if $F_{\gamma,\sigma}$ values at two points $\mathbf{s}_1$ and $\mathbf{s}_2$ in $\mathcal{S}_{\mathbf{x}}$ are larger than $m - n_0/(2+2\gamma)$, then the distance between $\mathbf{s}_1$ and $\mathbf{s}_2$ is bounded by $2\sqrt{m(\gamma+1)}\sigma$.

Lemma 2: Let $F = F_{\gamma,\sigma}$ where $\gamma = \gamma(n_0)$. If for two points $\mathbf{s}_1$ and $\mathbf{s}_2$ of $\mathcal{S}_{\mathbf{x}}$ we have:

$$F(\mathbf{s}_i) \geq m - \frac{n_0}{2+2\gamma}, \quad i = 1, 2, \tag{27}$$

then:

$$\|\mathbf{s}_1 - \mathbf{s}_2\| \leq 2\sqrt{m(\gamma+1)}\sigma. \tag{28}$$

Moreover, if $\mathbf{s}_2 = \mathbf{s}_0$, we have a slightly stricter bound

$$\|\mathbf{s}_1 - \mathbf{s}_0\| \leq \sqrt{m(\gamma+1)}\sigma\cdot \tag{29}$$

Proof: The argument is similar to that of Lemma 1 of [17], but made a bit more rigorous. Having in mind the proof of the first part of Corollary 1, observe that (27) implies that $\mathbf{s}_1$ and $\mathbf{s}_2$ have at most $n_0/2$ elements with absolute values greater than σ. Hence, $\mathbf{s}_1 - \mathbf{s}_2$ has at most n_0 elements with absolute values greater than 2σ. Let I index those elements of $\mathbf{s}_1 - \mathbf{s}_2$ with absolute values greater than 2σ. Then $|I| \leq n_0$ and

$$\|\pi_{I^c}(\mathbf{s}_1 - \mathbf{s}_2)\|^2 \leq |I^c|(2\sigma)^2 \leq 4m\sigma^2. \tag{30}$$

From (20) and (30), we get

$$\|\pi_I(\mathbf{s}_1 - \mathbf{s}_2)\|^2 \leq 4m\sigma^2\gamma \tag{31}$$

and

$$\|\mathbf{s}_1 - \mathbf{s}_2\|^2 \leq 4m\sigma^2(1+\gamma), \tag{32}$$

which yield (28). If $\mathbf{s}_2 = \mathbf{s}_0$, we can conclude that $\mathbf{s}_1 - \mathbf{s}_2$ has at most n_0 elements with absolute values greater than σ, and hence

$$\|\mathbf{s}_1 - \mathbf{s}_0\|^2 \leq m\sigma^2(1+\gamma). \tag{33}$$

■

D. Bounded variations of cost functions

Our cost functions have a nice property which ℓ^0 does not, *i.e.* they are continuous. In Lemma 3 we show that the derivative of f is bounded, and as a result, small changes in s result in small changes in $F(\mathbf{s})$.

Lemma 3: For $f = f_{\gamma,\sigma}$ and $F = F_{\gamma,\sigma}$:

$$|f'(s)| < \frac{2}{(1+\gamma)\sigma} \tag{34}$$

and

$$|F(\mathbf{s}_1) - F(\mathbf{s}_2)| \leq \frac{2\sqrt{m}}{(1+\gamma)\sigma}\|\mathbf{s}_1 - \mathbf{s}_2\| \tag{35}$$

for any $s \in \mathbb{R}$ and $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^m$.

Proof: (34) is a straight forward conclusion from (16). To prove (35), note that for any $\mathbf{s} \in \mathbb{R}^m$ we have

$$\|\nabla F(\mathbf{s})\|_2 = \sqrt{\sum_{i=1}^{m} |f'(s_i)|^2} \leq \frac{2\sqrt{m}}{(1+\gamma)\sigma}, \tag{36}$$

where ∇F denotes the gradient of $F : \mathbb{R}^m \to \mathbb{R}$. Moreover, using the mean value theorem, for any $\mathbf{s}_1$ and $\mathbf{s}_2$ there exists a $\mathbf{s} \in \mathbb{R}^m$ such that

$$F(\mathbf{s}_1) - F(\mathbf{s}_2) = \nabla F(\mathbf{s})^T(\mathbf{s}_1 - \mathbf{s}_2) \tag{37}$$

Therefore:

$$\begin{aligned} |F(\mathbf{s}_1) - F(\mathbf{s}_2)| &= |\nabla F(\mathbf{s})^T(\mathbf{s}_1 - \mathbf{s}_2)| \\ &\leq \|\nabla F(\mathbf{s})\|_2 \cdot \|\mathbf{s}_1 - \mathbf{s}_2\| \leq \frac{2\sqrt{m}}{(1+\gamma)\sigma}\|\mathbf{s}_1 - \mathbf{s}_2\| \cdot \end{aligned} \tag{38}$$

■

E. The choice of parameters of the algorithm

At this point we have acquired the necessary tools for designing a sequence of σ values needed to successfully maximize $F_{\gamma,\sigma}$. The question remained to be solved is how, after finding the global maximum of $F_{\gamma,\sigma}$ for some value of σ, we choose the next value of σ so that we are guaranteed to be in a (locally) concave area. More specifically, Lemma 2 ensures that by starting from any point s satisfying $F_{\gamma,\sigma}(\mathbf{s}) \geq m - n_0/(2+2\gamma)$ and following the steepest ascent trajectory of $F_{\gamma,\sigma}$, we end at the global maximum $\mathbf{s}_*$ of $F_{\gamma,\sigma}$ satisfying $F_{\gamma,\sigma}(\mathbf{s}_*) \geq m - k$. The question we study next is how, knowing $F_{\gamma,\sigma}(\mathbf{s}_*) \geq m-k$, can we choose the next value of σ' subject to $F_{\gamma,\sigma'}(\mathbf{s}_*) \geq m - n_0/(2+2\gamma)$. In Lemma 4 we present a constant c, for which $\sigma' = c\sigma$ satisfies this condition.

Lemma 4: For constants $B \geq A \geq 0$, let's define

$$c \triangleq \frac{2m}{2m+B-A}. \tag{39}$$

Then we have the following result:

$$\text{If } F_{\gamma,\sigma}(\mathbf{s}) \geq m - A, \text{ then } F_{\gamma,c\sigma}(\mathbf{s}) \geq m - B, \tag{40}$$

for any $\mathbf{s} \in \mathbb{R}^m$. Moreover

$$F_{\gamma,\sigma}(\mathbf{s}) \geq m - \frac{\|\mathbf{s}\|^2}{(1+\gamma)\sigma^2}. \tag{41}$$

Proof: For (41) note that:

$$f_\gamma(s/\sigma) \geq 1 - s^2/(1+\gamma)\sigma^2 \Rightarrow F_{\gamma,\sigma}(\mathbf{s}) \geq m - \frac{\|\mathbf{s}\|^2}{(1+\gamma)\sigma^2}.$$

Let's define:

$$\alpha(t) \triangleq F_{\gamma,\sigma/(1+t)}(\mathbf{s}) = F_{\gamma,\sigma}(\mathbf{s} + \mathbf{s}t) = \sum_{i=1}^{n} f_{\gamma,\sigma}(s_i + s_i t)$$

for $t \geq 0$. Having $|f'_{\gamma,\sigma}(s)| \leq 2/(1+\gamma)\sigma$ from (34), and $f'_{\gamma,\sigma}(s) = 0$ for $|s| \geq (1+\gamma)\sigma$ from (16), we will have

$$\begin{aligned} |\frac{d}{dt}\alpha(t)| &= |\sum_{i=1}^{m} \frac{d}{dt} f_{\gamma,\sigma}(s_i + s_i t)| \leq \sum_{i=1}^{m} |s_i| \cdot |f'_{\gamma,\sigma}(s_i + s_i t))| \\ &= \sum_{|s_i| < \sigma(1+\gamma)} |s_i| \cdot |f'_{\gamma,\sigma}(s_i + s_i t))| \leq 2m \cdot \end{aligned}$$

Hence, by choosing $t_0 = (B-A)/(2m)$, we have

$$|\alpha(t_0) - \alpha(0)| \leq t_0 |\frac{d}{dt}\alpha(t)| \leq B - A$$

for some $t \geq 0$. Then, choosing $c = 1/(1+t_0)$ in (39), we have

$$|F_{c\sigma}(\mathbf{s}) - F_\sigma(\mathbf{s})| = |\alpha(t_0) - \alpha(0)| \leq B - A, \tag{42}$$

which leads to (40). ■

Using Lemma 4, the following theorem states a sufficient condition for the convergence of an asymptotic version of SL0, in which the steepest ascent follows exactly the steepest ascent trajectory (*i.e.* the case $\mu \to 0$ and $L \to \infty$).

Theorem 2: Assume $\mathbf{A}$ satisfies the URP and f is as defined in (13), and also $k \triangleq \|\mathbf{s}_0\|_0 < n_0/(2+2\gamma)$. Let $\hat{\mathbf{s}} \triangleq \operatorname{argmin}_{\mathbf{A}\mathbf{s}=\mathbf{x}} \|\mathbf{s}\| = \mathbf{A}^\dagger \mathbf{x}$ and:

$$\sigma_1 = \frac{\|\hat{\mathbf{s}}\|}{\sqrt{k(1+\gamma)}} \tag{43}$$

$$c = \frac{2m}{2m + n_0/(2+2\gamma) - k} < 1 \cdot \tag{44}$$

If we choose the geometric sequence of σ according to $\sigma_{j+1} = c\sigma_j$, and set $\mathbf{s}_1 = \hat{\mathbf{s}}$ in the first step, and in each subsequent step, *i.e.* $j \geq 2$, start with $\mathbf{s}_{j-1}$ and move on the steepest ascent trajectory of F_{σ_j} to reach the maximizer $\mathbf{s}_j$, then at each step:

$$F_{\sigma_j}(\mathbf{s}_j) \geq m - k$$

and

$$\lim_{j \to \infty} \mathbf{s}_j = \mathbf{s}_0.$$

Proof: By induction on j. First note that by substituting σ_1 defined by (43) in (41), we have $F_{\sigma_1}(\mathbf{s}_1) = F_{\sigma_1}(\hat{\mathbf{s}}) \geq m-k$. Moreover, by substituting c defined by (44) in Lemma 4 we conclude

$$F_\sigma(\mathbf{s}) \geq m-k \Rightarrow F_{c\sigma}(\mathbf{s}) \geq m - \frac{n_0}{2+2\gamma}, \qquad (45)$$

for any $\mathbf{s} \in \mathbb{R}^m$. Now, to complete the induction, assume $F_{\sigma_{j-1}}(\mathbf{s}_{j-1}) \geq m-k$. Then, from (45)

$$F_{\sigma_j}(\mathbf{s}_{j-1}) = F_{c\sigma_{j-1}}(\mathbf{s}_{j-1}) \geq m - \frac{n_0}{2+2\gamma}\cdot \qquad (46)$$

Therefore, according to Lemma 2, the $\mathbf{s}_j$ which is achieved by starting at $\mathbf{s}_{j-1}$ and following the steepest ascent trajectory of F_{σ_j}, satisfies $F_{\sigma_j}(\mathbf{s}_j) \geq m-k$.

To prove the second part of Theorem 2, note that $\sigma_j \to 0$ as $j \to \infty$ (since $c < 1$) and $m-k \geq m-n+k$ (since $k \leq n_0/2 \leq n/2$), hence the sequence of $\mathbf{s}_j$ satisfies the conditions of Theorem 1. The same conclusion also follows from Lemma 2, since $F_\sigma(\mathbf{s}_j) \geq m-k \geq m-n_0/(2+2\gamma)$ results in

$$\|\mathbf{s}_j - \mathbf{s}_0\| \leq \sqrt{m(\gamma+1)}\sigma_j \to 0\cdot \qquad (47)$$

■

Remark 1. Theorem 2 proves the convergence of an asymptotic version of SL0, in which the internal loop steps precisely along the steepest ascent trajectory. This corresponds to $\mu \to 0$ and $L \to \infty$ in Fig. 1. We will discuss later in Section IV the case of $\mu > 0$ (discrete steps in the steepest ascent directions), and propose a value for μ which guarantees the convergence, provided that the internal loop is repeated until the convergence is achieved (corresponding to $L \to \infty$). Finally, Section VI proposes a value for L that guarantees the convergence and that completes the convergence analysis of SL0.

Remark 2. In [17] (Remark 5, section III) we heuristically justified that σ_1 should be chosen proportional to the maximum absolute value of elements of $\mathbf{s}$, *i.e.* $\max_i |s_i|$. This choice is now better justified by Theorem 2, Eq. (43).

Remark 3. In Experiment 2 of [17] we had observed that the value of c depended on the sparsity (k) of the solution, and not as much on any other parameter of SL0 (see Fig. 3 of [17]). The optimal value of c grew with increasing k and tended to 1 as $k \to n/2$. Equation (44) supports this observation, as the value of c depends only on the value of k (and of course, the system scale), and $c \to 1$ as $k \to n_0/2(1+\gamma)$.

Corollary 3: Asymptotic SL0 (when $\mu \to 0$ and $L \to \infty$) converges to the sparse solution if

$$\alpha\delta^{\min}_{\lceil 2k\alpha \rceil} + \|\mathbf{A}\|_2 \leq \alpha, \qquad (48)$$

where $k = \|\mathbf{s}\|_0$, $\alpha > 1$ is an arbitrary constant, $\delta^{\min}_k$ is the lower ARIC, and $\|\mathbf{A}\|_2$ denotes the spectral norm of $\mathbf{A}$.

Proof: If (48) holds, by setting $n_0 = \lceil 2\alpha k \rceil$ and using (12), it is easy to see that $\gamma(n_0)+1 < \alpha$. Hence, the condition of Theorem 2, *i.e.* $\|\mathbf{s}\|_0 < n_0/(2+2\gamma(n_0))$, holds and the convergence is guaranteed.

■

III. Large Random Gaussian Matricies

Our sparsity constraint for successful recovery of the sparse solution is of the form $k < n_0/(2+2\gamma)$, where $\gamma = \gamma(n_0)$ depends on the matrix $\mathbf{A}$. It is not practical to precisely calculate $\gamma(n_0)$ for large scale systems since computational complexity grows exponentially[5]. However, in the case of random Gaussian matrices we can find reasonable almost sure (a.s.) upper-bounds on $\gamma(n_0)$, which make it possible to compare our results with the ones for ℓ^1-minimization [28], [23], [24], [25]. In this section we assume that $\mathbf{A}$ has independent identically distributed (i.i.d) entries drawn from a normal distribution with zero mean and variance $1/n$.

We use Theorem II.13 of [27]. Let $\mathbf{G}$ be an $l \times n$ random matrix with i.i.d. entries drawn from a $N(0, 1/n)$ distribution. We are interested in singular values of $\mathbf{G}$, or equivalently, eigenvalues of $\mathbf{G}^T\mathbf{G}$, and, in particular, the smallest and the largest one. In [27], [26], authors prove that

$$\mathbb{P}\left\{\sigma_{\max}(\mathbf{G}) > 1 + \sqrt{l/n} + r\right\} \leq \exp(-nr^2/2) \qquad (49)$$

and also

$$\mathbb{P}\left\{\sigma_{\min}(\mathbf{G}) < 1 - \sqrt{l/n} - r\right\} \leq \exp(-nr^2/2)\cdot \qquad (50)$$

They prove the above inequalities for the case $l \leq n$. It is not difficult to check that (49) holds for the case $l > n$ as well, since from definition of $\mathbf{G}$, $\sqrt{n/l}\,\mathbf{G}^T$ is an $n \times l$ normal distributed matrix with variance $1/l$. In this case, we can use (49) to conclude that

$$\mathbb{P}\left\{\sigma_{\max}(\sqrt{n/l}\,\mathbf{G}^T) > 1 + \sqrt{n/l} + r\right\} \leq \exp(-lr^2/2)\cdot$$

Noting that $\sigma_{\max}(\sqrt{n/l}\,\mathbf{G}^T) = \sqrt{n/l}\,\sigma_{\max}(\mathbf{G})$ and setting $r' = r\sqrt{l/n}$ we get the desired result.

In the following theorem, using arguments similar to ones used for bounding the symmetric and asymmetric RICs [23], [24], [25], [20], we prove that with high probability the value of γ is bounded.

Theorem 3: If $\mathbf{A}$ is a random Gaussian matrix with i.i.d. zero mean entries of variance $1/n$ and if $\alpha = n/m$ and $\beta = n_0/m$ are fixed, then

$$\mathbb{P}\left\{\gamma(n_0) > \frac{(1+\sqrt{1/\alpha}+\epsilon)^2}{(1-\sqrt{\beta/\alpha}-r)^2}\right\} \leq \exp(-nr^2/2 + nr_0^2/2) + \exp(-n\epsilon^2/2), \qquad (51)$$

which tends to zero as $m \to \infty$, provided that $\epsilon > 0$ and $r > r_0$ where

$$r_0 \triangleq \sqrt{2\beta/\alpha \log(\boldsymbol{e}/\beta)} \qquad (52)$$

and $\boldsymbol{e} = \exp(1)$ denotes the Euler's constant (the base of natural logarithm).

Proof: Let I be some subset of $\{1, \cdots, m\}$ with $|I| = n_0$. Then, $\mathbf{A}_I$ is $n_0 \times n$ and

$$\mathbb{P}\left\{\sigma_{\max}(\mathbf{A}) > 1 + \sqrt{m/n} + \epsilon\right\} \leq \exp(-n\epsilon^2/2) \qquad (53)$$

[5]Even a deterministic upper bound on γ using (12) is not practical. The upper bound depends on Euclidean norm of $\mathbf{A}$ and the lower ARIC. Precise calculaion of ARIC requires enumerating all possible n_0-column submatrices of $\mathbf{A}$ and computing their smallest singular values.

and

$$\mathbb{P}\left\{\sigma_{\min}(\mathbf{A}_I) < 1 - \sqrt{n_0/n} - r\right\} \le \exp(-nr^2/2) \quad (54)$$

for any subset $|I| = n_0$. There are a total of $\binom{m}{n_0}$ such subsets, which means

$$\mathbb{P}\left\{\min_{|I|=n_0} \sigma_{\min}(\mathbf{A}_I) < 1 - \sqrt{n_0/n} - r\right\} \le \binom{m}{n_0} \boldsymbol{e}^{-nr^2/2}. \quad (55)$$

Then, using (11) we have

$$\mathbb{P}\left\{\gamma(n_0) > \frac{(1+\sqrt{m/n}+\epsilon)^2}{(1-\sqrt{n_0/n}-r)^2}\right\} \le \binom{m}{n_0}\exp(-nr^2/2) + \exp(-n\epsilon^2/2)\cdot \quad (56)$$

From

$$\binom{m}{n_0} \le \left(\frac{m\boldsymbol{e}}{n_0}\right)^{n_0} \le \exp\Big(n_0\log(m\boldsymbol{e}/n_0)\Big) \quad (57)$$

we get

$$\mathbb{P}\left\{\gamma(n_0) > \frac{(1+\sqrt{m/n}+\epsilon)^2}{(1-\sqrt{n_0/n}-r)^2}\right\} \le \exp\Big(n_0\log(m\boldsymbol{e}/n_0) - nr^2/2\Big) + \exp(-n\epsilon^2/2)\cdot \quad (58)$$

If we assume $\alpha = n/m$ and $\beta = n_0/m$ are fixed, then by defining r_0 as in (52), we obtain (51) as $m \to \infty$. ■

Corollary 4: Let's define $\gamma(\alpha,\beta)$ as follows:

$$\gamma(\alpha,\beta) \triangleq \frac{(1+\sqrt{1/\alpha})^2}{\Big(1-\sqrt{\beta/\alpha}-\sqrt{2\beta/\alpha\log(\boldsymbol{e}/\beta)}\Big)^2},$$

if $1 - \sqrt{\beta/\alpha} - \sqrt{2\beta/\alpha\log(\boldsymbol{e}/\beta)} > 0$, and otherwise $\gamma(\alpha,\beta) \to +\infty$. Let also

$$\rho(\alpha) \triangleq \max_{0\le\beta\le\alpha} \frac{\beta}{2+2\gamma(\alpha,\beta)}. \quad (59)$$

Then, $\rho(\alpha) > 0$ for any $\alpha > 0$. Moreover, we can guarantee that for almost every large system with ratio $n/m \to \alpha$, the asymptotic SL0 can recover the sparse solutions satisfying $\|\mathbf{s}\|_0 \le \rho(\alpha)m$.

Proof: To show $\rho(\alpha) > 0$, simply note that

$$\lim_{\beta\to 0^+} \frac{\beta}{2+2\gamma(\alpha,\beta)} = 0^+. \quad (60)$$

For the second part, it suffice to apply Theorem 3 with $n_0 = \lceil \beta^* m \rceil$, where β^* is the value of β that maximizes $\gamma(\alpha,\beta)$ in (59). ■

IV. Stability of the internal loop and its exponential convergence rate

From Fig. 1, the steepest ascent steps in SL0 are of the form:

$$\mathbf{s}_{i+1} = \mathbf{s}_i + \mu\sigma^2\mathbf{D}^T\mathbf{D}\nabla F|_{\mathbf{s}_i} \quad (61)$$

where $\mathbf{D}^T\mathbf{D}$ is the orthogonal projection on null$(\mathbf{A})$ and μ is the step size parameter. Until now, we have considered convergence of what we refer to as asymptotic version of SL0 (corresponding to $\mu \to 0$ and $L \to \infty$), in which the steps of the internal loop of Fig. 1 follow exactly along the steepest ascent trajectory . In this section, we study how to choose the parameter μ. For this part of the analysis, we assume the internal loop is repeated until convergence (corresponding to $L \to \infty$).

Lemma 5: Let $F = F_{\gamma',\sigma}$, where $\gamma' > \gamma = \gamma(n_0)$ and $\sigma > 0$ is arbitrary. Let also $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest and largest eigenvalues of $-\mathbf{D}\sigma^2\mathbf{H}_F(\mathbf{s})\mathbf{D}^T$ respectively (note that the values of $\lambda_{\min}$ and $\lambda_{\max}$ depend on $\mathbf{s}$). Then for all $\mathbf{s} \in \mathbb{R}^m$

$$\lambda_{\max} \le \frac{2}{1+\gamma} \quad (62)$$

and for all $\mathbf{s} \in \mathcal{A}$

$$\lambda_{\min} \ge \frac{2(\gamma'-\gamma)}{(1+\gamma)(\gamma'+\gamma'^2)}. \quad (63)$$

Proof: For convenience, let's define

$$\lambda'_{\max} \triangleq \frac{2}{1+\gamma}, \quad \lambda'_{\min} \triangleq \frac{2(\gamma'-\gamma)}{(1+\gamma)(\gamma'+\gamma'^2)}, \quad (64)$$

so that we need to show that

$$\lambda_{\max} \le \lambda'_{\max}, \quad \lambda_{\min} \ge \lambda'_{\min}. \quad (65)$$

We know that for any matrix $\mathbf{M}$ with maximum and minimum eigenvalues $\lambda_{\max}(\mathbf{M})$ and $\lambda_{\min}(\mathbf{M})$, $\mathbf{M} - \lambda\mathbf{I}$ is positive semi-definite if and only if $\lambda \le \lambda_{\min}(\mathbf{M})$. Moreover $\mathbf{M} - \lambda\mathbf{I}$ is negative semi-definite if and only if $\lambda \ge \lambda_{\max}$.

To prove (65), we show that $\mathbf{D}(\sigma^2\mathbf{H}_F(\mathbf{s}) + \lambda'_{\max}\mathbf{I})\mathbf{D}^T$ is positive semi-definite for all $\mathbf{s} \in \mathbb{R}^m$, and $\mathbf{D}(\sigma^2\mathbf{H}_F(\mathbf{s}) + \lambda'_{\min}\mathbf{I})\mathbf{D}^T$ is negative semi-definite as long as $\mathbf{s} \in \mathcal{A}$. Following steps of the proof of Lemma 1, the former follows from

$$\mathbf{w}^T\Big(\sigma^2\mathbf{H}_F(\mathbf{s}) + \lambda'_{\max}\mathbf{I}\Big)\mathbf{w} \ge (\lambda'_{\max} - \frac{2}{1+\gamma})\|\mathbf{w}\|^2 \ge 0. \quad (66)$$

To show the second assertion, from (20) we obtain $\|\mathbf{w}_I\|^2/\|\mathbf{w}_I^c\|^2 \le \gamma$. Then, from (19) we have

$$\mathbf{w}^T\Big(\sigma^2\mathbf{H}_F(\mathbf{s}) + \lambda'_{\min}\mathbf{I}\Big)\mathbf{w} \le (\lambda'_{\min} - \frac{2}{1+\gamma})\|\mathbf{w}_{I^c}\|^2 + (\lambda'_{\min} + \frac{2}{\gamma^2+\gamma})\|\mathbf{w}_I\|^2 \le 0\cdot \quad (67)$$

Hence, $\mathbf{D}(\sigma^2\mathbf{H}_F(\mathbf{s}) + \lambda'_{\min}\mathbf{I})\mathbf{D}^T$ and $\mathbf{D}(\sigma^2\mathbf{H}_F(\mathbf{s}) + \lambda'_{\max}\mathbf{I})\mathbf{D}^T$ are negative and positive semi-definite respectively, and (65) holds. ■

Theorem 4: Let $F = F_{\gamma',\sigma}$, where $\gamma' > \gamma = \gamma(n_0)$. Suppose also that:

$$F(\mathbf{s}_i) \ge m - \frac{n_0}{2+2\gamma}. \quad (68)$$

Then, by setting

$$\mu = 2/(\lambda'_{\min} + \lambda'_{\max}), \quad (69)$$

where $\lambda'_{\max}$ and $\lambda'_{\min}$ are as defined in (64), it is guaranteed that

$$\|\mathbf{s}_{i+1} - \mathbf{s}_{opt}\| \le \mathrm{CR}'\|\mathbf{s}_i - \mathbf{s}_{opt}\|, \quad (70)$$

where $\mathbf{s}_{opt}$ is the maximizer of F on $\mathcal{S}_\mathbf{x}$, $\mathbf{s}_{i+1}$ is as defined in (61), and $\mathrm{CR}' \triangleq (\lambda'_{\max} - \lambda'_{\min})/(\lambda'_{\max} + \lambda'_{\min})$ determines the convergence rate. Moreover:

$$F(\mathbf{s}_{i+1}) \geq F(\mathbf{s}_i). \tag{71}$$

Proof: The proof consists of the following steps.

Step 1: From (68), $\mathbf{s}_i \in \mathcal{A}$ and $\mathbf{s}_{opt} \in \mathcal{A}$, where $\mathcal{A}$ is as defined in Corollary 1. From Corollary 1, $\mathcal{A}$ is convex and F is concave on $\mathcal{A}$. Hence, $\mathbf{s}_{opt}$ satisfies:

$$\mathbf{s}_{opt} = \mathbf{s}_{opt} + \mu\sigma^2 \mathbf{D}^T \mathbf{D} \nabla F|_{\mathbf{s}_{opt}} \Leftrightarrow \mathbf{D}\nabla F|_{\mathbf{s}_{opt}} = 0. \tag{72}$$

Subtracting (72) from (61), we have

$$\mathbf{s}_{i+1} - \mathbf{s}_{opt} = \mathbf{s}_i - \mathbf{s}_{opt} + \mu\sigma^2 \mathbf{D}^T\mathbf{D}(\nabla F|_{\mathbf{s}_i} - \nabla F|_{\mathbf{s}_{opt}}). \tag{73}$$

Multiplying by $\mathbf{D}$ and setting $\mathbf{D}\mathbf{D}^T = \mathbf{I}$, we get

$$\mathbf{D}(\mathbf{s}_{i+1} - \mathbf{s}_{opt}) = \mathbf{D}(\mathbf{s}_i - \mathbf{s}_{opt}) + \mu\sigma^2 \mathbf{D}(\nabla F|_{\mathbf{s}_i} - \nabla F|_{\mathbf{s}_{opt}}), \tag{74}$$

From the mean value theorem, there exists a $t \in [0, 1]$ such that $\mathbf{s}' \triangleq t\mathbf{s}_{opt} + (1-t)\mathbf{s}_i$ satisfies:

$$\mathbf{D}(\mathbf{s}_{i+1} - \mathbf{s}_{opt}) = \mathbf{D}(\mathbf{s}_i - \mathbf{s}_{opt}) + \mu\sigma^2 \mathbf{D}\mathbf{H}_F(\mathbf{s}')(\mathbf{s}_i - \mathbf{s}_{opt}). \tag{75}$$

Since $\{\mathbf{s}_i, \mathbf{s}_{opt}\} \in \mathcal{A}$, it means that $\mathbf{s}' \in \mathcal{A}$. Also, since $(\mathbf{s}_i - \mathbf{s}_{opt}) \in \mathrm{null}(\mathbf{A})$, it is equal to its projection to null($\mathbf{A}$), that is, $\mathbf{s}_i - \mathbf{s}_{opt} = \mathbf{D}^T\mathbf{D}(\mathbf{s}_i - \mathbf{s}_{opt})$. Therefore, the above equation can be written as

$$\mathbf{D}(\mathbf{s}_{i+1} - \mathbf{s}_{opt}) = \Big(\mathbf{I} + \mu\sigma^2 \mathbf{D}\mathbf{H}_F(\mathbf{s}')\mathbf{D}^T\Big)\mathbf{D}(\mathbf{s}_i - \mathbf{s}_{opt}). \tag{76}$$

Since $(\mathbf{s}_i - \mathbf{s}_{opt})$ and $(\mathbf{s}_{i+1} - \mathbf{s}_{opt})$ are both in null($\mathbf{A}$), from (6) we can write

$$\|\mathbf{s}_{i+1} - \mathbf{s}_{opt}\| \leq \|\mathbf{I} + \mu\sigma^2\mathbf{D}\mathbf{H}_F(\mathbf{s}')\mathbf{D}^T\|_2 \cdot \|\mathbf{s}_i - \mathbf{s}_{opt}\|. \tag{77}$$

Step 2: Let's define the Rate of Convergence (CR) as

$$\begin{aligned}\mathrm{CR} &= \|\mathbf{I} + \mu\sigma^2\mathbf{D}\mathbf{H}_F(\mathbf{s}')\mathbf{D}^T\|_2 \\ &= \max\{|1-\mu\lambda_{\min}|, |1-\mu\lambda_{\max}|\} \\ &= \max\{1-\mu\lambda_{\min}, -1+\mu\lambda_{\min}, -1+\mu\lambda_{\max}, 1-\mu\lambda_{\max}\} \\ &= \max\{1-\mu\lambda_{\min}, -1+\mu\lambda_{\max}\},\end{aligned} \tag{78}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of $-\mathbf{D}\sigma^2\mathbf{H}_F(\mathbf{s}')\mathbf{D}^T$. The value of μ that optimizes CR is $\mu = 2/(\lambda_{\max} + \lambda_{\min})$, which results in

$$\mathrm{CR} = 1 - 2\frac{\lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} = \frac{\kappa - 1}{\kappa + 1}, \tag{79}$$

where $\kappa = \kappa(-\sigma^2\mathbf{D}\mathbf{H}_F(\mathbf{s}')\mathbf{D}^T) = \lambda_{\max}/\lambda_{\min}$ denotes the condition number of matrix $\mathbf{D}$. With this definition, we have

$$\|\mathbf{s}_{i+1} - \mathbf{s}_{opt}\| \leq \mathrm{CR}\|\mathbf{s}_i - \mathbf{s}_{opt}\|. \tag{80}$$

Computing $\lambda_{\max}$ and $\lambda_{\min}$ in each step is not practical for large scale systems. Instead, we can find bounds on their values using (65). Of course this bounds do not depend on $\mathbf{s}'$.

Choosing μ according to (69) and considering (78), we have

$$\|\mathbf{I} + \mu\sigma^2\mathbf{D}\mathbf{H}_F(\mathbf{s}')\mathbf{D}^T\|_2 \leq \frac{\lambda'_{\max} - \lambda'_{\min}}{\lambda'_{\max} + \lambda'_{\min}} = \mathrm{CR}'. \tag{81}$$

Taking (81) together with (77), we obtain (70).

Step 3: From the second order Taylor expansion of F around $\mathbf{s}_i$ we have

$$\begin{aligned}F(\mathbf{s}_{i+1}) - F(\mathbf{s}_i) &= (\mathbf{s}_{i+1} - \mathbf{s}_i)^T\nabla F \\ &+ \frac{1}{2}(\mathbf{s}_{i+1} - \mathbf{s}_i)^T\mathbf{H}_F(\mathbf{s}_{i+1} - \mathbf{s}_i)\end{aligned} \tag{82}$$

where $\nabla F = \nabla F|_{\mathbf{s}_i}$ and $\mathbf{H}_F = \mathbf{H}_F(\mathbf{s}'')$, for some point $\mathbf{s}''$ satisfying $\mathbf{s}'' = t\mathbf{s}_i + (1-t)\mathbf{s}_{i+1}$ for some $0 \leq t \leq 1$. Then, by substituting $\mathbf{s}_{i+1} - \mathbf{s}_i$ from (61) and factoring we get

$$\begin{aligned}&F(\mathbf{s}_{i+1}) - F(\mathbf{s}_i) = \\ &\quad \frac{\mu^2\sigma^2}{2}\nabla F^T\mathbf{D}^T\Big(\sigma^2\mathbf{D}\mathbf{H}_F\mathbf{D}^T + (2/\mu)\mathbf{I}\Big)\mathbf{D}\nabla F.\end{aligned} \tag{83}$$

From (69) and (65) we have

$$\lambda_{\max} \leq 2/\mu. \tag{84}$$

Now (71) is a straightforward conclusion of (83) and (84). ■

Remark 1. The value of $\gamma < \gamma' < (n_0/2k) - 1$ should be chosen carefully. If $\gamma' \to \gamma$, then $\lambda'_{\max}/\lambda'_{\min} \to \infty$ and $\mathrm{CR}' \to 1$. If $\gamma' \to (n_0/2k) - 1$, then $c \to 1$ in (44), and the computational cost tends to infinity. In Section VI, we discuss how to choose γ' to have a reasonable convergence.

Remark 2. Theorems 2 and 4 prove convergence of SL0, provided that the internal loop is repeated until convergence is reached. The question remains to be answered is how to select the value of L to guarantee that the internal loop is repeated until convergence is reached. This question is answered in Section VI.

V. The Noisy Case

Thus far we discussed the convergence and stability of SL0 in the noiseless case. Theorem 3 of [17] states that the maximizer of F_σ is a good estimator of sparse solution even in the noisy case. In this section we investigate the choice of parameters that assure local concavity and, hence, convergence of SL0 when data contains noise.

The following theorem is a modification of Theorem 3 of [17] and it provides conditions for convergence in noise.

Theorem 5: Let $\mathcal{S}_\epsilon = \{\mathbf{s} | \, \|\mathbf{A}\mathbf{s} - \mathbf{x}\| < \epsilon\}$, where ϵ is an arbitrary positive number, and assume that matrix $\mathbf{A}$ and functions f_σ satisfy the conditions of Theorem 2. Let $\mathbf{s}_0 \in \mathcal{S}_\epsilon$ be a sparse solution. Assume the condition $k < n_0/(2+2\gamma)$, and choose any k' satisfying $k < k' < n_0/(2+2\gamma)$. We also choose the first term σ_1 and the scale factor c according to

$$\sigma_1 = \frac{\|\hat{\mathbf{s}}\|}{\sqrt{k'(1+\gamma)}}, \tag{85}$$

$$c = \frac{2m}{2m + n_0/(2+2\gamma) - k'} < 1 \tag{86}$$

and set $\sigma_j = \sigma_1 c^{j-1}$, $1 \leq j \leq J$, where J is the index of the smallest term of the σ sequence satisfying

$$\sigma_J \geq \frac{2\sqrt{m}\|\mathbf{A}\|_2\epsilon}{(1+\gamma)(k'-k)} > \sigma_{J+1} = c\sigma_J. \tag{87}$$

Then, following the steps of asymptotic SL0 and terminating at step J, one can achieve a solution within the distance $C\epsilon$ of the sparsest solution, where

$$C = \Big(\frac{4m}{c(k'-k)\sqrt{\gamma+1}} + 1\Big)\|\mathbf{A}\|_2. \quad (88)$$

Proof: Let $\mathbf{n} \triangleq \mathbf{A}\mathbf{s}_0 - \mathbf{x}$. Then, $\mathbf{s}_0 \in \mathcal{S}_\epsilon$ means that $\|\mathbf{n}\| < \epsilon$. Defining $\tilde{\mathbf{n}} \triangleq \mathbf{A}^T\mathbf{n}$, we have

$$\mathbf{x} = \mathbf{A}\mathbf{s}_0 + \mathbf{n} = \mathbf{A}\mathbf{s}_0 + \mathbf{A}\mathbf{A}^T\mathbf{n} = \mathbf{A}\mathbf{s}_0 + \mathbf{A}\tilde{\mathbf{n}} = \mathbf{A}(\mathbf{s}_0 + \tilde{\mathbf{n}}) = \mathbf{A}\tilde{\mathbf{s}},$$

where $\tilde{\mathbf{s}} \triangleq \mathbf{s}_0 + \tilde{\mathbf{n}}$. Let $\mathbf{s}_\sigma$ be the maximizer of F_σ on $\mathbf{A}\mathbf{s} = \mathbf{x}$, as defined in Theorem 1 of [17]. Note that, $\mathbf{s}_\sigma$ is not necessarily the maximizer of F_σ on the whole $\mathcal{S}_\epsilon$. The argument is similar to that of Theorem 3 in [17]. From (35) in lemma 3 and (87), we have

$$\begin{aligned} &\|\tilde{\mathbf{s}} - \mathbf{s}_0\| = \|\tilde{\mathbf{n}}\| < \|\mathbf{A}\|_2\epsilon \Rightarrow \\ &|F_{\sigma_j}(\tilde{\mathbf{s}}) - F_{\sigma_j}(\mathbf{s}_0)| \le \frac{2\sqrt{m}}{\sigma_j(1+\gamma)}\|\tilde{\mathbf{s}} - \mathbf{s}_0\| \le k' - k\cdot \end{aligned} \quad (89)$$

Hence

$$F_{\sigma_j}(\mathbf{s}_0) \ge m - k \Rightarrow F_{\sigma_j}(\tilde{\mathbf{s}}) \ge m - k'\cdot \quad (90)$$

The vector $\mathbf{s}_0$ does not necessarily satisfy $\mathbf{A}\mathbf{s} = \mathbf{x}$, however, we have chosen $\tilde{\mathbf{s}}$ to be the projection of $\mathbf{s}_0$ onto the subspace $\mathbf{A}\mathbf{s} = \mathbf{x}$. Hence, $\tilde{\mathbf{s}}$ satisfies $\mathbf{A}\mathbf{s} = \mathbf{x}$. Moreover, $F_\sigma(\tilde{\mathbf{s}}) > m - k' > m - n_0/(2+2\gamma)$, hence $\tilde{\mathbf{s}} \in \mathcal{A}$, and by optimizing F_σ from an arbitrary point in $\mathcal{A}$ we are guaranteed a solution $\mathbf{s}_*$ for which $F_\sigma(\mathbf{s}_*) \ge m - k'$. Now, using Lemma 4, it is easy to conclude that for σ_1 and c chosen according to (85) and (86),

$$F_{\sigma_1}(\hat{\mathbf{s}}) \ge m - k' \quad (91)$$

and

$$F_\sigma(\mathbf{s}) \ge m - k' \Rightarrow F_{c\sigma}(\mathbf{s}) \ge m - n_0/(2+2\gamma). \quad (92)$$

Following the steps of the proof of Theorem 2, but with the sparsity factor k replaced by k', we can conclude that

$$F_{\sigma_J}(\mathbf{s}_J) \ge m - k'\cdot \quad (93)$$

Using Lemma 2, (90) and (93), we then have

$$\|\mathbf{s}_J - \tilde{\mathbf{s}}\| \le 2\sqrt{m(\gamma+1)}\sigma_J \le \frac{4m\|\mathbf{A}\|_2\epsilon}{c\sqrt{1+\gamma}(k'-k)} \quad (94)$$

and

$$\begin{aligned} \|\mathbf{s}_J - \mathbf{s}_0\| &\le \|\mathbf{s}_J - \tilde{\mathbf{s}}\| + \|\tilde{\mathbf{s}} - \mathbf{s}_0\| \\ &\le \frac{4m\|\mathbf{A}\|_2\epsilon}{c\sqrt{1+\gamma}(k'-k)} + \|\mathbf{A}\|_2\epsilon = C\epsilon\cdot \end{aligned} \quad (95)$$

■

Remark 1. If $k' \to k$, the error bound tends to infinity in (88). If $k' \to n_0/(2+2\gamma)$, the computational cost would tend to infinity as $c \to 1$ in (86). Hence k' should be chosen suitably between the two values. A simple sub-optimal choice is presented in the next section.

Remark 2. In Theorem 3 of [17], we proved that by suitably choosing σ proportional to the noise level, we can bound the Euclidean distance between the maximizer of F_σ and the sparse solution by order of the noise standard deviation. Experiment 2 of [17] (Section IV, Fig. 4) confirmed the result of Theorem 3 of [17]. Here, (87) and (88) also confirm this result. As can be seen from (88), the estimation error depends linearly on the system noise.

VI. FINALIZING THE CONVERGENCE ANALYSIS

At this point we have acquired all the tools necessary for ensuring the convergence of the external loop, stability of the steepest ascent (internal loop), and robustness against noise for SL0. The only parameter we have not yet discussed is L (the number of iterations of the internal loop shown in Fig. 1). In this section, we put all the previous results together and provide values for all the parameters that are sufficient to guarantee successful convergence of SL0.

We present results for three cases. In the first case, we assume that suitable values of n_0 and $\gamma = \gamma(n_0)$ are known, such that $\|\mathbf{s}_0\|_0 < n_0/(2+2\gamma)$. In this case, the values of the parameters that guarantee the convergence are summarized in Fig. 2 and the convergence is proved in Theorem 6.

In the second case, γ is assumed unknown and we consider a large Gaussian matrix $\mathbf{A}$, and use the almost sure results of Section III to determine n_0 and γ. The values of the parameters for this case that guarantee convergence are summarized in Fig. 3. For a random matrix $\mathbf{A}$ with i.i.d and zero-mean Gaussian entries, Theorem 7 shows that using these parameters the sparse solution of $\mathbf{A}\mathbf{s} = \mathbf{x}$ can be found with probability approaching 1 as the size of the system grows, as long as $\|\mathbf{s}_0\|_0 < \rho(\alpha)m$. Moreover, it is shown that the complexity of SL0 grows as m^2, which is faster than the state of the art $m^{3.5}$ associated with Basic Pursuit and is comparable with Matching Pursuit.

The third case deals with multiple source recovery where the sparsest solutions of multiple USLE's with the same coefficient matrix are recovered at once. Multiple source recovery may be viewed in the context of SCA [4] for Blind Source Separation. In Experiment 6 of [17] we observed that implementing SL0 for multiple source recovery in matrix multiplication form can make it faster than the SL0 algorithm for single solution recovery. Theorem 8 shows that this approach can speed up SL0 to the order of $m^{1.376}$.

A. Case of known γ

Putting the results of previous sections together, the following theorem shows that if the values of the parameters are chosen as summarized in Fig. 2, then SL0 will converge to the sparsest solution. The proposed value for L can be seen in the step 17 of the figure. Note also that the notation of Fig. 1 has been changed slightly in Fig. 2 to match the convergence proof given next.

Theorem 6 (The case of known n_0 and γ): Let $\gamma = \gamma(n_0)$ and, without loss of generality, assume matrix $\mathbf{A}$ has orthonormal rows. Let $\mathbf{x} = \mathbf{A}\mathbf{s}_0 + \mathbf{n}$ for some $\|\mathbf{n}\| \le \epsilon$ and $\|\mathbf{s}_0\|_0 \le k < n_0/2(1+\gamma)$. Let $\Delta \triangleq \frac{n_0/2(1+\gamma)-k}{4m}$. Then the algorithm given in Fig. 2 can recover $\mathbf{s}_0$ within a distance $\delta > C\epsilon$, where

$$C \triangleq \Big(\frac{4}{\Delta\sqrt{\gamma+1}} + 1\Big)\|\mathbf{A}\|_2\cdot \quad (96)$$

- Initialization:
  1) $\Delta \leftarrow \frac{n_0/2(1+\gamma)-k}{4m}$
  2) $k' \leftarrow k + m\Delta$
  3) $k'' \leftarrow k + 2m\Delta$
  4) $\frac{n_0}{2(1+\gamma')} \leftarrow k + 3m\Delta$ (*i.e.* $\gamma' \leftarrow \frac{n_0}{2(k+3m\Delta)} - 1$)
  5) $F \leftarrow F_{\gamma'}$
  6) $\delta' \leftarrow \delta - \|\mathbf{A}\|_2\epsilon$
  7) $\sigma_1 \leftarrow \|\mathbf{A}^T\mathbf{x}\|/\sqrt{n_0/(2+2\gamma')}$
  8) $\sigma_J \leftarrow \delta'/2\sqrt{m(\gamma'+1)}$
  9) $J \leftarrow \lceil \frac{\log(\sigma_1)-\log(\sigma_J)}{\log(1+\Delta/2)} \rceil + 1$
  10) $\log(c) \leftarrow -\frac{\log(\sigma_1)-\log(\sigma_J)}{J-1}$
  11) $\sigma_j \leftarrow \sigma_1 c^{j-1} (1 \le j \le J)$
  12) $\lambda'_{\max} \leftarrow \frac{2}{1+\gamma}$
  13) $\lambda'_{\min} = \frac{2(\gamma'-\gamma)}{(1+\gamma)(\gamma'^2+\gamma')}$
  14) $\mu \leftarrow 2/(\lambda'_{\min} + \lambda'_{\max})$
  15) $\kappa' \leftarrow \lambda'_{\max}/\lambda'_{\min}$
  16) $\mathrm{CR}' \leftarrow \frac{\kappa'-1}{\kappa'+1}$
  17) $L \leftarrow \lceil \frac{-\log(\Delta/4)-1/2\log(\gamma'+1)}{-\log(\mathrm{CR}')} \rceil + 1$
- For $j = 1, \ldots, J$:
  1) $\sigma \leftarrow \sigma_j$.
  2) If $j \ge 2$, $\mathbf{s}_{j,1} \leftarrow \mathbf{s}_{j-1,L}$. If $j = 1$, $\mathbf{s}_{1,1} \leftarrow \mathbf{A}^T\mathbf{x}$
  3) For $l = 1, \ldots, L-1$:
     – $\mathbf{s}_{j,l+1} \leftarrow \mathbf{s}_{j,l} + \mu\sigma^2\mathbf{D}^T\mathbf{D}\nabla F_\sigma|_{\mathbf{s}_{j,l}}$
- Output is $\mathbf{s}_{out} \leftarrow \mathbf{s}_{J,L}$.

Fig. 2. The SL0 algorithm for the case of known n_0 and $\gamma(n_0)$ and $\mathbf{A}$ with orthonormal rows , with parameters shown that guarantee convergence to the sparsest solution. $\mathbf{s}_{j,l}$ is the solution estimate at the corresponding iteration.

Proof: The proof is constructed using the following steps:

Step 1: Let's set $\tilde{\mathbf{s}} = \mathbf{s}_0 + \mathbf{A}^T\mathbf{n}$, then we have $\tilde{\mathbf{s}} \in \mathcal{S}_\mathbf{x}$ and also $F_\sigma(\tilde{\mathbf{s}}) \ge m - k'$ for any $\sigma \ge \sigma_J$. Assume that we have

$$\sigma \ge \sigma_J = \frac{\delta'}{2\sqrt{m(\gamma'+1)}}\cdot \tag{97}$$

Then, from Lemma 3 we have

$$|F_\sigma(\mathbf{s}_0) - F_\sigma(\tilde{\mathbf{s}})| \le \frac{2\sqrt{m}}{(1+\gamma')\sigma}\|\mathbf{s}_0 - \tilde{\mathbf{s}}\|\cdot \tag{98}$$

Since $\|\mathbf{s}_0 - \tilde{\mathbf{s}}\| \le \|\mathbf{A}^T\|_2 \cdot \|\mathbf{n}\| \le \|\mathbf{A}\|_2\epsilon$ and

$$\delta' \triangleq \delta - \|\mathbf{A}\|_2\epsilon \ge \frac{4}{\sqrt{\gamma+1}\Delta}\|\mathbf{A}\|_2\epsilon, \tag{99}$$

from (97), (98) and (99) we have

$$|F_\sigma(\mathbf{s}_0) - F_\sigma(\tilde{\mathbf{s}})| \le m\Delta \Rightarrow F_\sigma(\tilde{\mathbf{s}}) \ge F_\sigma(\mathbf{s}_0) - m\Delta = m - k'. \tag{100}$$

Step 2: We show that for any $1 \le j \le J$, if $F_{\sigma_j}(\mathbf{s}_{j,1}) \ge m - \frac{n_0}{2+2\gamma'}$, then $F_{\sigma_j}(\mathbf{s}_{j,L}) \ge m - k''$, where the notations $\mathbf{s}_{j,i}$ and k'' are defined in Fig. 2. Let $\mathbf{s}_{opt}$ be the maximizer of F_{σ_j} on $\mathcal{S}_\mathbf{x}$. Hence, $F_{\sigma_j}(\mathbf{s}_{opt}) \ge F_{\sigma_j}(\mathbf{s}_{j,1}) \ge m - \frac{n_0}{2+2\gamma'}$ and from Lemma 2,

$$\|\mathbf{s}_{j,1} - \mathbf{s}_{opt}\| \le 2\sqrt{m(\gamma+1)}\sigma. \tag{101}$$

From (70), we conclude

$$\begin{aligned}\|\mathbf{s}_{j,L} - \mathbf{s}_{opt}\| &\le (\mathrm{CR}')^{L-1}\|\mathbf{s}_{j,1} - \mathbf{s}_{opt}\| \\ &\le \frac{\Delta\sqrt{\gamma'+1}}{4}\left(2\sqrt{m(\gamma'+1)}\sigma_j\right) \\ &\le \frac{\sqrt{m}\Delta\sigma_j(1+\gamma')}{2},\end{aligned} \tag{102}$$

where the second inequality holds when the value of L is defined as in the Step 17 of Fig. 2. Hence, from Lemma 3

$$|F_{\sigma_j}(\mathbf{s}_{opt}) - F_{\sigma_j}(\mathbf{s}_{j,L})| \le \frac{2\sqrt{m}}{\sigma_j(1+\gamma')}\|\mathbf{s}_{j,L} - \mathbf{s}_{opt}\| = m\Delta. \tag{103}$$

Therefore, from (100) and (103) we have

$$F_{\sigma_j}(\mathbf{s}_{j,L}) \ge F_{\sigma_j}(\mathbf{s}_{opt}) - m\Delta \ge F_{\sigma_j}(\tilde{\mathbf{s}}) - m\Delta \ge m - k''. \tag{104}$$

Step 3: We show that if $F_{\sigma_{j-1}}(\mathbf{s}_{j-1,L}) \ge m - k''$, then

$$F_{\sigma_j}(\mathbf{s}_{j,1}) \ge m - n_0/(2+2\gamma')\cdot \tag{105}$$

From the algorithm of Fig. 2, we know that

$$c \ge \frac{1}{1+\Delta/2} = \frac{2m}{2m+m\Delta}\cdot \tag{106}$$

Then, choosing $A = k'' = k + 2m\Delta$ and $B = n_0/(2+2\gamma') = k + 3m\Delta$ in Lemma 4 and substituting $\mathbf{s}_{j,1} = \mathbf{s}_{j-1,L}$, we have

$$F_{\sigma_{j-1}}(\mathbf{s}_{j-1,L}) \ge m - k'' \Rightarrow F_{\sigma_j}(\mathbf{s}_{j,1}) \ge m - n_0/(2+2\gamma')\cdot \tag{107}$$

Step 4: Here, we prove by induction on j that $F_{\sigma_j}(\mathbf{s}_{j,L}) \ge m - k''$. In the first step, we have $\mathbf{s}_{1,1} = \mathbf{A}^T\mathbf{x}$ and

$$\sigma_1 = \|\mathbf{A}^T\mathbf{x}\|/\sqrt{n_0/(2+2\gamma')}\cdot \tag{108}$$

Hence, from Lemma 4

$$F_{\sigma_1}(\mathbf{s}_{1,1}) \ge m - n_0/(2+2\gamma'), \tag{109}$$

and from Step 2

$$F_{\sigma_1}(\mathbf{s}_{1,L}) \ge m - k''\cdot \tag{110}$$

Assume that

$$F_{\sigma_{j-1}}(\mathbf{s}_{j-1,L}) \ge m - k'' \tag{111}$$

for some j. Then from the results of Step 3 and noting from Fig 2 that $\mathbf{s}_{j,1} = \mathbf{s}_{j-1,L}$, we obtain

$$F_{\sigma_j}(\mathbf{s}_{j,1}) = F_{\sigma_j}(\mathbf{s}_{j-1,L}) \ge m - n_0/(2+2\gamma') \tag{112}$$

and from Step 2,

$$F_{\sigma_j}(\mathbf{s}_{j,L}) \ge m - k''. \tag{113}$$

We can conclude then that

$$F_{\sigma_J}(\mathbf{s}_{out}) = F_{\sigma_J}(\mathbf{s}_{J,L}) \ge m - k'' \ge m - n_0/(2+2\gamma'). \tag{114}$$

Step 5: From Lemma 2, (114), (100) and the choice of σ_J given in step 8 of Fig. 2, we have

$$\|\hat{\mathbf{s}} - \mathbf{s}_{out}\| \le 2\sqrt{m(\gamma'+1)}\sigma_J = \delta' \tag{115}$$

and

$$\|\mathbf{s}_0 - \mathbf{s}_{out}\| \le \|\hat{\mathbf{s}} - \mathbf{s}_{out}\| + \|\mathbf{s}_0 - \hat{\mathbf{s}}\| \le \delta' + \|\mathbf{A}\|_2\epsilon = \delta\cdot \tag{116}$$

This completes the proof of convergence of SL0. ∎

Remark 1. In noiseless case ($\epsilon = 0$), SL0 can recover the sparsest solution within a distance δ, for some $\delta > 0$, in a finite number of steps. But as $\delta \to 0$, σ_J, i.e. the last value of σ, tends to zero according to step 8 of Fig. 2, and J tends to ∞ according to step 9. Hence, the complexity of the algorithm tends to infinity.

Remark 2. Note that the algorithm does not require the exact value of the ℓ^0 norm. Only an upper bound k is necessary.

- Initialization:
  1) $\beta^* \leftarrow$ maximizer of $\beta/(2+2\gamma(\alpha,\beta))$ on $0 \le \beta \le \alpha$
  2) $\gamma \leftarrow \gamma(\alpha,\beta^*)$
  3) $n_0 \leftarrow \lceil \beta^* m \rceil$
  4) $k \leftarrow \lceil rm \rceil$
  5) $\delta \leftarrow C'\epsilon$, where C' is defined in (117)
  6) $\sigma_1 \leftarrow (1+\sqrt{\alpha})(1+\sqrt{\alpha}+\epsilon)$. (This step replaces step 7 of Fig. 2).
  7) Do initialization steps $1\cdots 6$ and $8\cdots 17$ of Fig. 2.

Fig. 3. SL0 initialization parameters for the case of unknown γ. Step 6 here replaces step 7 of Fig. 2.

B. Case of unknown γ

For a large Gaussian $\mathbf{A}$, we can use the a.s. results of Section III to find n_0 and $\gamma(n_0)$, and thus obtain the initialization of SL0 shown in Fig. 3. The following theorem guarantees convergence of the algorithm in Fig. 3.

Theorem 7 (the case of unknown n_0 and γ): Let $\mathbf{A}$ be an $n \times m$ Gaussian matrix, and $n/m \to \alpha > 0$ as $m \to \infty$. Lets fix $r < \rho(\alpha)$ and let $\mathbb{P}_m$ denote the probability that the algorithm in Fig. 3 can recover any $\mathbf{s}_0$ from $\mathbf{x} = \mathbf{A}\mathbf{s}_0 + \mathbf{n}$ within Euclidean distance of $\delta = C'\epsilon$, as long as $\|\mathbf{s}_0\|_0 < rm$, $\|\mathbf{s}_0\| \le 1$, and $\|\mathbf{n}\| < \epsilon$, where

$$C' \triangleq \left(\frac{16}{\Big(\rho(\alpha)-r\Big)\sqrt{\gamma+1}} + 1\right)(1+\sqrt{\alpha}). \tag{117}$$

Then, we have $\mathbb{P}_m \to 1$ as $m \to \infty$. Moreover, the complexity of the algorithm is $O(m^2)$.

Proof: We know from Theorem 3 that $\mathbb{P}\left\{\gamma(n_0) > \gamma\right\} \to 0$ as $m \to \infty$. Moreover, $\mathbb{P}\left\{\|\mathbf{A}\|_2 > \sqrt{\alpha}+1\right\} \to 1$ as $m \to \infty$ [27], [26]. Therefore noting $\mathbf{x} = \mathbf{A}\mathbf{s}_0 + \mathbf{n}$ we have

$$\mathbb{P}\left\{\|\mathbf{A}^T\mathbf{x}\| \le (1+\sqrt{\alpha})^2 + (1+\sqrt{\alpha})\epsilon\right\} \to 1 \tag{118}$$

as $m \to \infty$, because $\|\mathbf{s}_0\|_2 \le 1$ and $\|\mathbf{n}\| < \epsilon$. This means that the condition imposed by step 6 of Fig. 3 is stricter than that imposed by step 7 of Fig. 2. Thus, all the conditions of Theorem 6 also apply for the algorithm in Fig. 3. Hence, the Euclidean distance between the final solution and the sparsest solution is less than $C\epsilon$, *i.e.*

$$\|\mathbf{s}_{out} - \mathbf{s}_0\|_2 < C\epsilon \tag{119}$$

where C is as defined in (96). Moreover, $\mathbb{P}\left\{C < C'\right\} \to 1$ as $m \to \infty$, where C' is as defined in (117). Hence, the accuracy is better than $C'\epsilon$ with probability tending to 1, which completes the proof of the convergence result.

From Fig. 2, it is clear that the computational complexity of SL0 is $O(mnJL)$ and since $n/m \to \alpha > 0$, we can assume $n = O(m)$. To obtain the final complexity result, we show that $J = O(1)$ and $L = O(1)$ as $m \to \infty$. According to Fig. 2,

$$J < \frac{\log(\sigma_1/\sigma_J)}{\log(1+\Delta/2)} + 2. \tag{120}$$

From the initialization of Δ shown in Fig. 2,

$$\lim_{m\to\infty} \Delta > \frac{\beta^*/(2+2\gamma) - r}{4} = \frac{\rho(\alpha)-r}{4} > 0 \tag{121}$$

and

$$\lim_{m\to\infty} \log(1+\Delta/2) > 0\cdot \tag{122}$$

Hence to show that $J = O(1)$, we need to show that

$$\lim_{m\to\infty} \sqrt{m}\sigma_1 < \infty \tag{123}$$

and

$$\lim_{m\to\infty} \sqrt{m}\sigma_J > 0\cdot \tag{124}$$

To show (123) note that

$$\lim_{m\to\infty} \frac{\sqrt{m}}{\sqrt{n_0/(2+2\gamma')}} = \lim_{m\to\infty} \frac{1}{\sqrt{(k+3m\Delta)/m}} \le \frac{2}{\sqrt{\rho(\alpha)+3r}} < \frac{1}{\sqrt{r}}\cdot \tag{125}$$

With σ_1 given in Fig. 3, (123) becomes an obvious conclusion of (118) and (125). To show (124), note that from Fig. 2, we obtain

$$\lim_{m\to\infty} \sqrt{m}\sigma_J = (\delta'/2) \lim_{m\to\infty} \sqrt{1/(1+\gamma')} > (\delta'/2)\sqrt{3/4(1+\gamma)} > 0, \tag{126}$$

where we have used the fact that

$$n_0/(1+\gamma') = 3n_0/4(1+\gamma) + k/4 \Rightarrow 1/(1+\gamma') > 3/4(1+\gamma). \tag{127}$$

Next, we show that $L = O(1)$. Note that from Fig. 2

$$L < \frac{-\log(\Delta/4)}{-\log(\mathrm{CR}')} + 2. \tag{128}$$

From (121) we know that $-\log(\Delta/4)$ is bounded. Hence, to complete the proof of $L = O(1)$, we need to show that

$$\lim_{m\to\infty} \log(\mathrm{CR}') < 0 \Leftrightarrow \lim_{m\to\infty} \mathrm{CR}' < 1. \tag{129}$$

From the definition of $\lambda'_{\min}$, $\lambda'_{\max}$, and κ' in Fig. 2,

$$\kappa' = (\gamma'^2 + \gamma')/(\gamma' - \gamma). \tag{130}$$

Observe that

$$\gamma' - \gamma = \frac{n_0}{2(k+3m\Delta)} - \frac{n_0}{2(k+4m\Delta)} = \frac{n_0 m\Delta}{2(k+3m\Delta)(k+4m\Delta)} \tag{131}$$

and

$$\lim_{m\to\infty} \frac{\gamma'-\gamma}{1+\gamma'} = \lim_{m\to\infty} \frac{m\Delta}{k+4m\Delta} = \lim_{m\to\infty} \frac{\Delta}{k/m+4\Delta} > \frac{\Delta}{4\Delta+r} = \frac{\Delta}{\rho(\alpha)} > 0. \tag{132}$$

Also note that from (127), we have

$$\gamma' < 4/3(1+\gamma) - 1. \tag{133}$$

Then, from (132) and (133) one can conclude

$$\lim_{m\to\infty} \kappa' < \infty \tag{134}$$

and

$$\lim_{m\to\infty} \mathrm{CR}' = 1 - 2\lim_{m\to\infty} \frac{1}{\kappa'+1} < 1. \tag{135}$$

■

Remark 1. In [17], we experimentally observed that the optimal value of L is a small constant (Fig. 5, Experiment 2, section IV). Here, we proved that L is bounded as $m \to \infty$.

- Initialization: repeat initialization steps $1 \cdots 17$ of Fig. 2
- For $j = 1, \ldots, J$:
  1) $\sigma \leftarrow \sigma_j$.
  2) If $j \geq 2$, $\mathbf{S}_{j,1} \leftarrow \mathbf{S}_{j-1,L}$. If $j = 1$, $\mathbf{S}_{1,1} \leftarrow \mathbf{A}^T\mathbf{X}$
  3) For $l = 1, \ldots, L-1$:
     – $\mathbf{S}_{j,l+1} \leftarrow \mathbf{S}_{j,l} + \mu\sigma^2\mathbf{D}^T\mathbf{D}\nabla F_\sigma|_{\mathbf{S}_{j,l}}$
- Output is $\mathbf{S}_{out} \leftarrow \mathbf{S}_{J,L}$.

Fig. 4. MSL0 (SL0 for multiple sparse recovery). $\mathbf{S}_{j,i}$ is our estimation of the matrix of sparse solutions at the corresponding level.

C. Multiple Sparse Solution Recovery Case

Thus far, we discussed the recovery of the sparsest solution of USLE containing a single measurement vector. In SCA applications one deals with multiple measurement vectors.

The resulting system of equations can be written in matrix form:

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{N}, \tag{136}$$

where $\mathbf{X} \triangleq [\mathbf{x}(1), \ldots, \mathbf{x}(T)] \in \mathbb{R}^{n\times T}$, $\mathbf{S} \triangleq [\mathbf{s}(1), \ldots, \mathbf{s}(T)] \in \mathbb{R}^{m\times T}$ and $\mathbf{N} \triangleq [\mathbf{n}(1), \ldots, \mathbf{n}(T)] \in \mathbb{R}^{n\times T}$. As observed in Experiment 6 of [17], when we apply the MSL0 (SL0 for multiple sparse recovery) of Fig. 4, the overall computational complexity reduces as compared to T separate applications of the vector version of SL0. The following theorem supports this observation.

Theorem 8: Under the conditions of Theorem 7, using the algorithm shown in Fig. 4 to recover the sparsest solutions satisfying (136) reduces average computational complexity of each individual solution $\mathbf{x}(t), 1 \leq t \leq T$ to $O(m^{1.376})$ as $T/m \to \infty$.

Proof: Note that the only computationally expensive part of the algorithm is step 3 of the loop in Fig. 4, where we multiply the $m \times m$ matrix $\mathbf{D}^T\mathbf{D}$ by the $m \times 1$ vector $\nabla F_\sigma|_{\mathbf{s}_{j,l}}$, and also the initialization of $\mathbf{s}_{1,1}$, where we compute $\mathbf{A}^T\mathbf{x}$. This is because these two steps are of order $O(m^2)$, and all the other computations are at most of order $O(m)$. Analogous to approach of Experiment 6 in [17], we use the matrix form (136). We replace the final loop with steps shown in Fig. 4, and perform $m \times T$ matrix multiplication using $\lceil T/m \rceil$ multiplications of $m \times m$ matrices using Coppersmith-Winograd algorithm [34]. The overall complexity is T/m times $O(m^{2.376})$, or equivalently, T times $O(m^{1.376})$, meaning that per sample complexity is $O(m^{1.376})$. ■

VII. Conclusion

We had recently proposed the SL0 algorithm, which we showed empirically to be efficient and accurate for recovery of sparse solutions using ℓ^0 minimization [17]. Its convergence properties, however, were only partially analyzed, so the theoretical justification for SL0 remained incomplete. The current paper provides the theoretical justification for SL0.

Several results were presented. First, general results were derived showing that a judicial choice of parameter values guarantees that SL0 converges to the sparsest solution provided that the given system satisfies the recovery conditions. These conditions were derived in terms of the lower asymmetric RIC and Eucleadian norm of the system. We then adapted the convergence results for the special case where the system is a large Gaussian matrix. Next, we showed that convergence of SL0 can be similarly guaranteed in the case of noise. The noise results combined with our previous work and numerical experiments presented in [17] indicate that SL0 exhibits good robustness properties in noise. Lastly, we provided the complete parameter setting of SL0, that guaranteed recovery of the sparsest solutions in the case of general as well as Gaussian system. We then extended the SL0 algorithm to the case of multiple measurement vectors and provided the necessary parameter settings for the convergence.

Also presented were computational complexity results for SL0 in the cases of single and multiple measurement vectors. We showed that in the limiting case $m \to \infty$ and $n/m \to \alpha > 0$, the complexity is $O(m^2)$ and is comparable to that of orthogonal MP techniques. Further, we showed that recovering multiple sparse solutions simultaneously by using MSL0 reduces complexity per individual solution to $O(m^{1.376})$.

The main purpose of the presented results is to fulfill the need for theoretical justification of SL0. A number of papers have stated that RIP provides a strict condition for analysis of sparse recovery algorithms and it typically leads to unnecessarily pessimistic choices for the theoretical parameter values. Our empirical findings in [17] confirm this assessment in the case of SL0 as well. We have observed fast convergence with excellent empirical recovery rates under weaker sufficient conditions than those that can be obtained from an ARIP analysis.

References

[1] E.J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information," *IEEE Transactions on Information Theory*, vol. 52, no. 2, pp. 489–509, February 2006.

[2] D. L. Donoho, "Compressed sensing," *IEEE Transactions on Information Theory*, vol. 52, no. 4, pp. 1289–1306, April 2006.

[3] R. G. Baraniuk, "Compressive sensing," *IEEE Signal Processing Magazine*, vol. 24, no. 4, pp. 118–124, July 2007.

[4] R. Gribonval and S. Lesage, "A survey of sparse component analysis for blind source separation: principles, perspectives, and new challenges," in *Proceedings of ESANN'06*, April 2006, pp. 323–330.

[5] P. Bofill and M. Zibulevsky, "Underdetermined blind source separation using sparse representations," *Signal Processing*, vol. 81, pp. 2353–2362, 2001.

[6] P. G. Georgiev, F. J. Theis, and A. Cichocki, "Blind source separation and sparse component analysis for over-complete mixtures," in *Proceedinds of ICASSP'04*, Montreal (Canada), May 2004, pp. 493–496.

[7] Y. Li, A. Cichocki, and S. Amari, "Sparse component analysis for blind source separation with less sensors than sources," in *ICA2003*, 2003, pp. 89–94.

[8] S. S. Chen, D. L. Donoho, and M. A. Saunders, "Atomic decomposition by basis pursuit," *SIAM Review*, vol. 43, no. 1, pp. 129–159, March 2001.

[9] D. L. Donoho, M. Elad, and V. Temlyakov, "Stable recovery of sparse overcomplete representations in the presence of noise," *IEEE Trans. Info. Theory*, vol. 52, no. 1, pp. 6–18, Jan 2006.

[10] E. J. Candès and T. Tao, "Decoding by linear programming," *IEEE Transactions on Information Theory*, vol. 51, no. 12, pp. 4203–4215, 2005.

[11] S. Mallat and Z. Zhang, "Matching pursuits with time-frequency dictionaries," *IEEE Trans. on Signal Proc.*, vol. 41, no. 12, pp. 3397–3415, 1993.

[12] I. F. Gorodnitsky and B. D. Rao, "Sparse signal reconstruction from limited data using FOCUSS, a re-weighted minimum norm algorithm," *IEEE Transactions on Signal Processing*, vol. 45, no. 3, pp. 600–616, March 1997.

[13] R. Gribonval and M. Nielsen, "Sparse decompositions in unions of bases," *IEEE Trans. Inform. Theory*, vol. 49, no. 12, pp. 3320–3325, Dec. 2003.

[14] D. L. Donoho and M. Elad, "Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ^1 minimization," *Proc. Nat. Aca. Sci.*, vol. 100, no. 5, pp. 2197–2202, March 2003.

[15] D. Donoho and J. Tanner, "Thresholds for the recovery of sparse solutions via l1 minimization," in *In Proceedings of the Conference on Information Sciences and Systems*, 2006.

[16] H. Mohimani, M. Babaie-Zadeh, and Ch. Jutten, "Fast sparse representation based on smoothed l0 norm," in *Proceedings of 7th International Conference on Independent Component Analysis and Signal Separation (ICA2007), Springer LNCS 4666*, London, UK, September 2007, pp. 389–396.

[17] H. Mohimani, M. Babaie-Zadeh, and Ch. Jutten, "A fast approach for overcomplete sparse decomposition based on smoothed ℓ^0 norm," *IEEE Transactions on Signal Processing*, vol. 57, no. 1, pp. 289–301, January 2009.

[18] A. Blake and A. Zisserman, *Visual Reconstruction*, MIT Press, 1987.

[19] M. Davies and R. Gribonval, "Restricted isometry constants where ℓ_p sparse recovery can fail for $0 < p \leq 1$," *IEEE Transactions of Information Theory*, 2009, to appear.

[20] J. Blanchard, C. Cartis, and J. Tanner, "Phase transitions for restricted isometry properties," in *Proceedings of* SPARS2009, Saint-Malo, France, 6–9 April 2009.

[21] S. Foucart and M. Lai, "Sparsest solutions of underdetermined linear systems via ℓ_q-minimization for $0 < q \leq 1$," *Applied and Computational Harmonic Analysis*, vol. 26, no. 3, pp. 395–407, 2009.

[22] C. Dossal, G. Peyre, and J. Fadili, "Challenging restricted isometry constants with greedy pursuit," in *Proceedings of IEEE Information Theory Workshop (ITW)*, Taormina, Italy, 11–16 October 2009.

[23] E. J. Candès and T. Tao, "Near-optimal signal recovery from random projections: Universal encoding strategies?," *IEEE Transactions on Information Theory*, vol. 52, no. 12, pp. 5406–5425, December 2006.

[24] E. Candès, T. Tao, and J. Romberg, "Stable signal recovery from incomplete and inaccurate measurements," *Communications On Pure And Applied Mathematics*, vol. 59, no. 8, pp. 1207–1223, 2006.

[25] E. Candès, "The restricted isometry property and its implications for compressed sensing," *Compte Rendus de l'Academie des Sciences, Paris, Serie I*, vol. 346, pp. 589–592, 2008.

[26] M. Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, Number 89, American Mathematical Society, 2001.

[27] K.R. Davidson and S.J. Szarek, "Local operator theory, random matrices and banach spaces," in *Handbook on the Geometry of Banach spaces*, W. B. Johnson and J. Lindenstrauss, Eds., vol. 1, pp. 317–366. Elsevier Science, 2001.

[28] D. L. Donoho, "For most large underdetermined systems of linear equations the minimal l^1-norm solution is also the sparsest solution," Tech. Rep., 2004.

[29] A. Nemirovskii I. E. Nesterov and Y. Nesterov, "Interior-point polynomial algorithms in convex programming," in *SIAM*, 1994.

[30] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms; Engineering Applications*, MPS-SIAM Series on Optimization, 2001.

[31] T. Blumensath and Davies M.E., "In greedy pursuit of new directions: (nearly) orthogonal matching pursuit by directional optimization,," in *European SIgnal Processing Conference (EUSIPCO'08)*, Lausanne, August 2008.

[32] S. Krstulovic and R. Gribonval, "MPTK: Matching pursuit made tractable," in *ICASSP'06*, Toulouse, France, May 2006, vol. 3, pp. 496–499.

[33] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1990.

[34] D. Coppersmith and S. Winograd, "Matrix multiplication via arithmetic progressions," *Journal of Symbolic Computation*, vol. 9, no. 3, pp. 251–280, March 1990.